ALTERNATIVE TO FACTOR-TYPE ESTIMATOR UNDER TWO-PHASE SAMPLING

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Abstract
In this paper alternative estimator to factor-type estimator for estimating population mean under two-phase sampling scheme has been proposed. The new estimator was obtained by transforming the means of both the study and the auxiliary variables using information on $(N-n)$ population units yet to be drawn. The expressions for biases and MSEs of the proposed estimator in the form of population parameters using the concept of large Sample approximation when secondary sample S of size n is a subset from the preliminary large sample S_1 of size n_1 and as an independent sample from population i.e. $S \subset \Omega$ have been derived and the conditions for its efficiency over the conventional estimators have been established. Also empirical study to demonstrate the efficiencies of the proposed estimator over traditional estimators have been performed and the results show that the proposed estimator performed better.

Keywords: Auxiliary variable, Factor-Type estimator, Two-phase Sampling, Mean Squared Error (MSE)

Introduction

The utilization of auxiliary information either at planning stage or at design stage to improve the estimate of population parameters has been a common phenomenon in the sample surveys. Auxiliary information can improved the efficiencies of estimators if they are judiciously utilized especially during estimation stage. The use of auxiliary information for formulating factor estimator for estimating population mean $\bar{Y}$ was introduced by Singh and Shukla (1987) and (1993) under single phase sampling and extended to two-phase sampling by Shukla (2002). The use of factor estimators fully depends on the information provided by the sample of size n and $n_1$drawn from the population of size N. In many situations the information provided by the sample drawn may be insufficient, bias or not representative of population of interest. In such situations information on $(N-n)$ population units yet to be drawn from both study and auxiliary variables is used for the estimation as it contains the adequate and representative information required.

In this paper alternative estimator for factor-type estimator which uses information on $(N-n)$ population units yet to be drawn from both study and auxiliary variables to transform both $\bar{x}$ and $\bar{y}$ has been proposed and its efficiency over conventional estimators has been theoretically and empirically demonstrated.

Consider a preliminary large sample S_1 of size n_1 drawn from population ? of size N by simple random sampling without replacement and secondary sample S of size $n(n < n_1)$ drawn either of the following manners:

Case I: as a subset from the preliminary sample i.e. $S \subset S_1$.

Case II: as an independent sample from population i.e. $S \subset \Omega$.

Let $\bar{y}$, $\bar{x}$ be means of secondary sample of size n and $\bar{x}_1$ be mean of preliminary sample of size n_1. The traditional ratio estimator $\bar{y}_r$ was proposed by Cochran (1942) and dual to ratio estimator $\bar{y}_{st}$ was proposed by Strivenkataramana (1980). Under two-phase sampling, these estimators can be defined as y_r^d and y_{st}^d respectively and their MSEs for case I and II are defined as follows;

$$\bar{y}_r^d = \frac{\bar{y}}{\bar{x}}\bar{x}_1$$

$$(1)$$

$$MSE(\bar{y}_r^d)_I = \bar{Y}^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}C_x^2 - 2\frac{1-f^*}{n}\rho C_x C_y\right) \quad (2)$$

$$MSE(\bar{y}_r^d)_{II} = \bar{Y}^2\left(\frac{1-f}{n}C_y^2 - \left(\frac{1-f}{n} + \frac{1-f_1}{n_1}\right)C_x^2 - 2\frac{1-f}{n}\rho C_x C_y\right) \quad (3)$$

$$\bar{y}_{st}^d = \bar{y}\frac{N\bar{x}_1 - n\bar{x}}{(N-n)\bar{x}_1} \quad (4)$$

$$MSE(\bar{y}_{st}^d)_I = \bar{Y}^2\left(\frac{1-f}{n}C_y^2 + \left(\frac{n}{N-n}\right)^2\frac{1-f^*}{n}C_x^2 - 2\frac{n}{N-n}\frac{1-f^*}{n}\rho C_x C_y\right) \quad (5)$$

$$MSE(\bar{y}_{st}^d)_{II} = \bar{Y}^2\left(\frac{1-f}{n}C_y^2 + \left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n} + \frac{1-f_1}{n_1}\right)C_x^2 - 2\frac{n}{N-n}\frac{1-f}{n}\rho C_x C_y\right) \quad (6)$$

Where $f = \frac{n}{N}$, $f_1 = \frac{n_1}{N}$, $f^* = \frac{n}{n_1}$

Adewara (2015) proposed alternative to Kadilar (2014) ratio-estimator using the means of both the study and the auxiliary variables using information on $(N-n)$ population units yet to be drawn under single-phase sampling. The estimator proposed and the MSE are given below

$$\bar{y}_{aa} = \bar{y}^*\frac{\bar{X}^*}{\bar{x}^*} \quad (7)$$

Where $\bar{x}^* = \bar{X}\left(1-\left(\frac{n}{N-n}\right)\Lambda_{\bar{x}}\right)$, $\bar{y}^* = \bar{Y}\left(1-\left(\frac{n}{N-n}\right)\Lambda_{\bar{y}}\right)$, $\Lambda_{\bar{x}} = \frac{\bar{x}-\bar{X}}{\bar{X}}$, $\Lambda_y = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$ such that $|\Lambda_{\bar{x}}| < 1$, $|\Delta_{x}| < 1, |\Delta_y| < 1$

$$MSE(\bar{y}_{aa}) = \bar{Y}^2\left(\frac{n}{N-n}\right)^2\frac{1-f}{n}\left(C_y^2 + C_x^2 - 2\rho C_x C_y\right) \quad (8)$$

Adewara (2015) obtained in his work that alternative to Kadilar and Cingi (2004) ratio-estimator y_{aa} was more efficient than classical ratio estimator y_r if $0 < n < \frac{N}{2}$.

Factor-Type Estimator

Shukla (2002) suggested a factor-type estimator for population mean under two-phase sampling as

$$\bar{y}_{FTd} = \bar{y}\frac{(A+C)\bar{x}_1 + fB\bar{x}}{(A+fB)\bar{x}_1 + C\bar{x}} \quad (9)$$

$$MSE(\bar{y}_{FTd})_I = \bar{Y}^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}P^2C_x^2 + 2\frac{1-f^*}{n}P\rho C_x C_y\right) \quad (10)$$

$$MSE(y_{FTd})_{II} = \bar{Y}^2\left(\frac{1-f}{n}C_y^2 + \left(\frac{1-f}{n} + \frac{1-f_1}{n_1}\right)P^2C_x^2 + 2\frac{1-f}{n}P\rho C_x C_y\right) \quad (11)$$

Where $A = (d-1)(d-2)$, $B = (d-1)(d-4)$, and $C = (d-2)(d-3)(d-4)$,
$\theta_1 = \frac{A+C}{A+fB+C}$, $\theta_2 = \frac{fB}{A+fB+C}$, $\theta_3 = \frac{A+fB}{A+fB+C}$, $\theta_4 = \frac{C}{A+fB+C}$, $P = \theta_3 - \theta_1 = \theta_2 - \theta_4$
$0 < d < \infty$

Shukla (2003) obtained in his work that factor-type estimator y_{FT} was more efficient than classical ratio estimator $\bar{y}_r$ if $-\frac{2\rho C_y}{C_x} < P < 0$ under case I and if $-\frac{2\rho C_y}{C_x}(1+\delta)^{-1} < P < 0$ under case II where $\delta = \left(\frac{1-f_1}{n_1}\right)\left(\frac{1-f}{n}\right)^{-1}$.

Proposed Estimator

Let $\bar{x}_1$ be the mean of auxiliary variable X based on preliminary large sample S_1 of size n_1, $\bar{x}$ and $\bar{y}$ be the means of auxiliary variable X and study variable Y respectively based on the secondary sample S of size $n(n < n_1)$ drawn from population ? of size N by simple random sampling without replacement and let x^* and y^* be the means of X and Y yet to be drawn (Srivenkataramana and Srinath (1976)) i.e. the means corresponding to $(N-n)$ population units. Motivated by Adewara (2015), the new factor estimator is given by

$$y^*_{FTD} = y^*\frac{(A+C)\bar{x}_1^* + fB\bar{x}^*}{(A+fB)\bar{x}_1^* + C\bar{x}^*} \quad (12)$$

Where $\bar{X} = f\bar{x} + (1-f)\bar{x}^*$, $\bar{Y} = f\bar{y} + (1-f)\bar{y}^*$, $\bar{x} = \bar{X}(1+\Delta_x)$, $\bar{y} = \bar{Y}(1+\Delta_y)$,
$x^* = \bar{X}\left(1-\left(\frac{n}{N-n}\right)\Delta_{\bar{x}}\right)$, $x_1^* = \bar{X}\left(1-\left(\frac{n}{N-n}\right)\Delta_{\bar{x}_1}\right)$, $y^* = \bar{Y}\left(1-\left(\frac{n}{N-n}\right)\Delta_{\bar{y}}\right)$, $\Delta_{\bar{x}} = \frac{\bar{x}-\bar{X}}{\bar{X}}$,
$\Delta_{\bar{x}_1} = \frac{\bar{x}_1-\bar{X}}{\bar{X}}$, $\Delta_y = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$ such that $|\Delta_{\bar{x}}| < 1$, $|\Delta_{\bar{x}_1}| < 1$, $|\Delta_{\bar{y}}| < 1$.

Bias and MSE of the Proposed Estimator y^*_{FTD}

Equation (12) can be expressed in terms of Δ_X and Δ_y as

$$\bar{y}^{*}_{FTD} = \bar{Y}\left(1 - \frac{n}{N-n}\Delta_{\bar{y}}\right)\frac{(A+C)\bar{X}\left(1-\left(\frac{n}{N-n}\right)\Delta_{\bar{x}_1}\right)+fB\bar{X}\left(1-\frac{n}{N-n}\Delta_{\bar{x}}\right)}{(A+fB)\bar{X}\left(1-\left(\frac{n}{N-n}\right)\Delta_{\bar{x}_1}\right)+C\bar{X}\left(1-\frac{n}{N-n}\Delta_{\bar{x}}\right)} \quad (13)$$

Using power series expansion, the simplification of equation (13) up to first order approximation $O(n^{-1})$ is given by

$$\bar{y}^{*}_{FTD} = \bar{Y}\left(1 - \frac{n}{N-n}\Delta_{\bar{y}}\right)\frac{1-\frac{(A+C)\left(\frac{n}{N-n}\Delta_{\bar{x}_1}\right)}{A+C+fB}-\frac{fB\left(\frac{n}{N-n}\Delta_{\bar{x}}\right)}{A+C+fB}}{1-\frac{(A+fB)\left(\frac{n}{N-n}\Delta_{\bar{x}_1}\right)}{A+C+fB}-\frac{C\left(\frac{n}{N-n}\Delta_{\bar{x}}\right)}{A+C+fB}}$$

$$\bar{y}^{*}_{FTD} = \bar{Y}\left(1 - \frac{n}{N-n}\Delta_{\bar{y}}\right)\left(1-\theta_1\frac{n}{N-n}\Delta_{\bar{x}_1} - \theta_2\frac{n}{N-n}\Delta_{\bar{x}}\right)\left(1-\theta_3\frac{n}{N-n}\Delta_{\bar{x}_1} - \theta_4\frac{n}{N-n}\Delta_{\bar{x}}\right)^{-1}$$

$$\bar{y}^{*}_{FTD} - \bar{Y} = \bar{Y}\Bigg((\theta_2-\theta_4)\left(\frac{n}{N-n}\right)^2\Delta_{\bar{y}}\Delta_{\bar{x}} - (\theta_3-\theta_1)\left(\frac{n}{N-n}\right)^2\Delta_{\bar{y}}\Delta_{\bar{x}_1} - \frac{n}{N-n}\Delta_{\bar{y}} - (\theta_2-\theta_4)\frac{n}{N-n}\Delta_{\bar{x}} + (\theta_3-\theta_1)\frac{n}{N-n}\Delta_{\bar{x}_1} - \theta_4(\theta_2-\theta_4)\left(\frac{n}{N-n}\right)^2\Delta^2_{\bar{x}} + \theta_3(\theta_3-\theta_1)\left(\frac{n}{N-n}\right)^2\Delta^2_{\bar{x}_1} + (2\theta_3\theta_4-\theta_1\theta_4-\theta_2\theta_3)\left(\frac{n}{N-n}\right)^2\Delta_{\bar{x}_1}\Delta_{\bar{x}}\Bigg)$$

$$\bar{y}^{*}_{FTD} - \bar{Y} = \bar{Y}\Bigg(P\left(\frac{n}{N-n}\right)^2\Delta_{\bar{y}}\Delta_{\bar{x}} - P\left(\frac{n}{N-n}\right)^2\Delta_{\bar{y}}\Delta_{\bar{x}_1} - \frac{n}{N-n}\Delta_{\bar{y}} - P\frac{n}{N-n}\Delta_{\bar{x}} + P\frac{n}{N-n}\Delta_{\bar{x}_1} - \theta_4 P\left(\frac{n}{N-n}\right)^2\Delta^2_{\bar{x}} + \theta_3 P\left(\frac{n}{N-n}\right)^2\Delta^2_{\bar{x}_1} + (\theta_4-\theta_3)P\left(\frac{n}{N-n}\right)^2\Delta_{\bar{x}_1}\Delta_{\bar{x}}\Bigg) \quad (14)$$

Under Case I

In case I, we have

$$\left.\begin{aligned} &E(\Delta_{\bar{y}}) = E(\Delta_{\bar{x}}) = E(\Delta_{\bar{x}_1}) = 0, E(\Delta_{\bar{x}_1})^2 = \frac{1-f_1}{n_1}C_x^2 \\ &E(\Delta_{\bar{y}})^2 = \frac{1-f}{n}C_y^2, E(\Delta_{\bar{x}})^2 = \frac{1-f}{n}C_x^2, E(\Delta_{\bar{y}}\Delta_{\bar{x}}) = \frac{1-f}{n}\rho C_y C_x \\ &E(\Delta_{\bar{y}}\Delta_{\bar{x}_1}) = \frac{1-f_1}{n_1}\rho C_y C_x, E(\Delta_{\bar{x}}\Delta_{\bar{x}_1}) = \frac{1-f_1}{n_1}C_x^2 \end{aligned}\right\} \quad (15)$$

Taking expectation of (14) and using results of (15), the bias of the proposed estimator to terms of order n^{-1} is obtained as

$$Bias(\bar{y}^{*}_{FTD})_I = \bar{Y}\frac{1-f^*}{n}\left(\frac{n}{N-n}\right)^2 P(\rho C_y C_x - \theta_4 C_x^2) \quad (16)$$

Squaring both sides of (14), then taking expectation and using results in (15), we obtain the MSE of the proposed estimator to terms of order n^{-1} as

$$MSE(\bar{y}^{*}_{FTD})_I = \bar{Y}^2\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}P^2C_x^2 + 2\frac{1-f^*}{n}P\rho C_x C_y\right) \quad (17)$$

$$MSE(\bar{y}^{*}_{FTD})_I = \left(\frac{n}{N-n}\right)^2 MSE(\bar{y}_{FTD})_I \quad (18)$$

Equations (10) and (18) are at minimum when $P = \theta_3 - \theta_1 = -\rho\frac{C_y}{C_x} = -v$ (say).

Simplifying relation $P = -v$, we obtained the cubic equation in the form of d as

$$(v-1)d^3 + (fv + f - 9v + 9)d^2 - (5fv + 5f - 27v + 26)d + (4fv + 4f - 25v + 24) = 0$$

(19)

By solving (19), at most 3 zeros d_1, d_2 and d_3 of the polynomials for which (9) and (12) are optimal will be obtained,

Under Case II

In case II, we have

$$\left.\begin{aligned} &E(\Delta_{\bar{y}}) = E(\Delta_{\bar{x}}) = E(\Delta_{\bar{x}_1}) = 0, E(\Delta_{\bar{x}_1})^2 = \frac{1-f_1}{n_1}C_x^2 \\ &E(\Delta_{\bar{y}})^2 = \frac{1-f}{n}C_y^2, E(\Delta_{\bar{x}})^2 = \frac{1-f}{n}C_x^2, E(\Delta_{\bar{y}}\Delta_{\bar{x}}) = \frac{1-f}{n}\rho C_y C_x \\ &E(\Delta_{\bar{y}}\Delta_{\bar{x}_1}) = 0, E(\Delta_{\bar{x}}\Delta_{\bar{x}_1}) = 0 \end{aligned}\right\} \quad (20)$$

Taking expectation of (14) and using results of (20), the bias of the proposed estimator to terms of order n^{-1} is obtained as

$$Bias(\bar{y}_{FTD})_{II} = \bar{Y}\left(\frac{n}{N-n}\right)^{2} P\left(\frac{1-f}{n}\rho C_y C_x + \left(\frac{1-f_1}{n}\theta_3 - \theta_1\frac{1-f}{n}\right)C_x^2\right) \quad (21)$$

Squaring both sides of (14), then taking expectation and using results in (20), we obtain the MSE of the proposed estimator to terms of order n^{-1} as

$$MSE(\bar{y}_{FTD})_{II} = \bar{Y}^2\left(\frac{n}{N-n}\right)^{2}\left(\frac{1-f}{n}C_y^2 + \left|\frac{1-f}{n} + \frac{1-f_1}{n_1}\right| P^2 C_x^2 + 2\frac{1-f}{n}P\rho C_x C_y\right) \quad (22)$$

$$MSE(\bar{y}^*_{FTD})_{II} = \left(\frac{n}{N-n}\right)^{2} MSE(\bar{y}_{FTD})_{II} \quad (23)$$

The MSEs of (9) and (20) is at minimum when $P = \theta_3 - \theta_2 = -\rho\frac{C_y}{C_x}(1+\delta)^{-1} = -k$ (say).

Simplifying relation $P = -k$, we obtained the cubic equation in the form of d as

$$(k-1)d^3 + (fk + f - 9k + 9)d^2 - (5fk + 5f - 27k + 26)d + (4fk + 4f - 25k - 24) = 0$$
(24)

By solving (24), at most 3 zeros d_1, d_2 and d_3 of the polynomials for which (9) and (12) are optimal will be obtained,

Efficiency Comparisons

In this section, efficiency of the proposed estimator is compared with that of sample mean, classical ratio estimator; factor-type estimator under case I and conditions are obtained under which the proposed estimator is more efficient.

Under Case I

(i) $Var(\bar{y}) - MSE(\bar{y}^*_{FTD})_I = Y^2\frac{1-f}{n}C_y^2 - Y^2\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}P^2C_x^2 + 2\frac{1-f^*}{n}P\rho C_y C_x\right) > 0$

$$\frac{1-f}{n}C_y^2 > \left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}P^2C_x^2 + 2\frac{1-f^*}{n}P\rho C_y C_x\right)$$

$$\frac{1-f}{n}\left(1-\left(\frac{n}{N-n}\right)^2\right)C_y^2 > \left(\frac{n}{N-n}\right)^2\frac{1-f^*}{n}\left(P^2C_x^2 + 2P\rho C_y C_x\right)$$

$$\frac{1-f}{1-f^*}\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2} > P(P+2v)$$

$$\left(\frac{n_1(N-n)(N-2n)}{n^2(n_1-n)}\right)\frac{C_y^2}{C_x^2} > P(P-2v)$$

$$\left(\frac{n_1(N-n)(N-2n)}{n^2(n_1-n)}\right)\frac{C_y^2}{C_x^2} - 2v < P < \left(\frac{n_1(N-n)(N-2n)}{n^2(n_1-n)}\right)\frac{C_y^2}{C_x^2}$$

Which is positive if $0 < n < \frac{N}{2}$ and $-2v < P < 0$.

(ii) $MSE(\bar{y}_r^d)_I - MSE(\bar{y}^*_{FTD})_I = Y^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}C_x^2 - 2\frac{1-f^*}{n}\rho C_x C_y\right) - Y^2\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}P^2C_x^2 + 2\frac{1-f^*}{n}P\rho C_y C_x\right) > 0$

$$\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}C_x^2 - 2\frac{1-f^*}{n}\rho C_x C_y\right) > \left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}P^2C_x^2 + 2\frac{1-f^*}{n}P\rho C_y C_x\right)$$

$$\frac{1-f}{n}\left(1-\left(\frac{n}{N-n}\right)^2\right)C_y^2 + \frac{1-f^*}{n}\left(C_x^2 - 2\rho C_x C_y\right) > \left(\frac{n}{N-n}\right)^2\frac{1-f^*}{n}\left(P^2C_x^2 + 2P\rho C_y C_x\right)$$

$$\frac{1-f}{1-f^*}\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2} + \left(\frac{N-n}{n}\right)^2(1-2v) > P(P+2v)$$

$$\left(\frac{n_1(N-n)(N-2n)}{n^2(n_1-n)}\right)\frac{C_y^2}{C_x^2} + \left(\frac{N-n}{n}\right)^2(1-2v) > P(P+2v)$$

$$\left(\frac{n_1(N-n)(N-2n)}{n^2(n_1-n)}\right)\frac{C_y^2}{C_x^2} + \left(\frac{N-n}{n}\right)^2\left(1-2v\left[1+\left(\frac{n}{N-n}\right)^2\right]\right) < P$$
$$< \left(\frac{n_1(N-n)(N-2n)}{n^2(n_1-n)}\right)\frac{C_y^2}{C_x^2} + \left(\frac{N-n}{n}\right)^2(1-2v)$$

Which is positive if $0 < n < \frac{N}{2}$ and $(1-4v) < P < (1-2v)$

(iii) $MSE(\bar{y}_{st}^{d})_{I} - MSE(\bar{y}_{PTD}^{*})_{I} = \bar{Y}^2 \frac{1-f}{n}\left(\frac{1-f}{n}C_y^2 + \left(\frac{n}{N-n}\right)^2 \frac{1-f^*}{n}C_x^2 - 2\frac{n}{N-n}\frac{1-f^*}{n}\rho C_x C_y\right) - \bar{Y}^2 \frac{1-f}{n}\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}P^2C_x^2 + 2\frac{1-f^*}{n}P\rho C_y C_x\right) > 0$

$$\left(\frac{1-f}{n}C_y^2 + \left(\frac{n}{N-n}\right)^2\frac{1-f^*}{n}C_x^2 - 2\frac{n}{N-n}\frac{1-f^*}{n}\rho C_x C_y\right) > \left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}P^2C_x^2 + 2\frac{1-f^*}{n}P\rho C_y C_x\right)$$

$$\frac{1-f}{n}\left(1-\left(\frac{n}{N-n}\right)^2\right)C_y^2 + \frac{1-f^*}{n}\left(\frac{n}{N-n}\right)^2\left(C_x^2 - 2\left(\frac{N-n}{n}\right)\rho C_x C_y\right) > \left(\frac{n}{N-n}\right)^2\frac{1-f^*}{n}\left(P^2C_x^2 + 2P\rho C_y C_x\right)$$

$$\frac{1-f}{1-f^*}\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2} + \left(1 - 2\left(\frac{N-n}{n}\right)v\right) > P(P+2v)$$

$$\left(\frac{n_1(N-n)(N-2n)}{n^2(n_1-n)}\right)\frac{C_y^2}{C_x^2} + \left(1 - 2\left(\frac{N-n}{n}\right)v\right) > P(P+2v)$$

$$\left(\frac{n_1(N-n)(N-2n)}{n^2(n_1-n)}\right)\frac{C_y^2}{C_x^2} + \left(1 - 2v\left[1+\left(\frac{N-n}{n}\right)\right]\right) < P < \left(\frac{n_1(N-n)(N-2n)}{n^2(n_1-n)}\right)\frac{C_y^2}{C_x^2} + \left(1 - 2\left(\frac{N-n}{n}\right)v\right)$$

Which is positive if $0 < n < \frac{N}{2}$ and $(1-4v) < P < (1-2v)$

(iv) $MSE(\bar{y}_{PTD})_{I} - MSE(\bar{y}_{PTD}^{*})_{I} = \bar{Y}^2 \frac{1-f}{n}\left(C_y^2 + P^2C_x^2 + 2P\rho C_y C_x\right) - \bar{Y}^2 \frac{1-f}{n}\left(\frac{n}{N-n}\right)^2\left(C_y^2 + P^2C_x^2 + 2P\rho C_y C_x\right) > 0$

$$1 > \left(\frac{n}{N-n}\right)^2$$

Which is positive if $0 < n < \frac{N}{2}$

Under Case II

(i) $Var(\bar{y}) - MSE(\bar{y}_{PTD}^{*})_{II} = \bar{Y}^2\frac{1-f}{n}C_y^2 - \bar{Y}^2\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \left[\frac{1-f}{n} + \frac{1-f_1}{n_1}\right]P^2C_x^2 + 2\frac{1-f}{n}P\rho C_x C_y\right) > 0$

$$\frac{1-f}{n}C_y^2 > \left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \left[\frac{1-f}{n} + \frac{1-f_1}{n_1}\right]P^2C_x^2 + 2\frac{1-f}{n}P\rho C_x C_y\right)$$

$$\left(1-\left(\frac{n}{N-n}\right)^2\right)C_y^2 > \left(\frac{n}{N-n}\right)^2\left((1+\delta)P^2C_x^2 + 2P\rho C_y C_x\right)$$

$$\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2} > P\left((1+\delta)P + 2v\right)$$

$$\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2} > P\left((1+\delta)P + 2v\right)$$

$$\left(\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2} - 2v\right)/(1+\delta) < P < \left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2}$$

Which is positive if $0 < n < \frac{N}{2}$ and $-2v/(1+\delta) < P < 0$

(ii) $MSE(\bar{y}_r^{d})_{II} - MSE(\bar{y}_{PTD}^{*})_{II} = \bar{Y}^2\left(\frac{1-f}{n}C_y^2 + \left(\frac{1-f}{n} + \frac{1-f_1}{n_1}\right)C_x^2 - 2\frac{1-f}{n}\rho C_x C_y\right) - \bar{Y}^2\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2 + \left[\frac{1-f}{n} + \frac{1-f_1}{n_1}\right]P^2C_x^2 + 2\frac{1-f}{n}P\rho C_x C_y\right) > 0$

$$\left(\frac{1-f}{n}C_y^2+\left(\frac{1-f}{n}+\frac{1-f_1}{n_1}\right)C_x^2-2\frac{1-f}{n}\rho C_xC_y\right)$$
$$>\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2+\left[\frac{1-f}{n}+\frac{1-f_1}{n_1}\right]P^2C_x^2+2\frac{1-f}{n}P\rho C_xC_y\right)$$
$$\left(1-\left(\frac{n}{N-n}\right)^2\right)C_y^2+\left((1+\delta)C_x^2-2\rho C_yC_x\right)>\left(\frac{n}{N-n}\right)^2\left((1+\delta)P^2C_x^2+2P\rho C_yC_x\right)$$
$$\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2}+\left(\frac{N-n}{n}\right)^2\left((1+\delta)-2v\right)>P\left((1+\delta)P+2v\right)$$
$$\left[\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2}+\left(\frac{N-n}{n}\right)^2\left((1+\delta)-2v\left[1+\left(\frac{n}{N-n}\right)^2\right]\right)\right]/(1+\delta)<P$$
$$<\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2}+\left(\frac{N-n}{n}\right)^2\left((1+\delta)-2v\right)$$

Which is positive if $0<n<\frac{N}{2}$ and $\left(1-\frac{4v}{(1+\delta)}\right)<P<\left((1+\delta)-2v\right)$

(iii) $MSE\left(\bar{y}_{st}^{d}\right)_{II}-MSE(\bar{y}_{FTD}^{+})_{II}=\bar{Y}^2\left(\frac{1-f}{n}C_y^2+\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}+\frac{1-f_1}{n_1}\right)C_x^2-2\frac{n}{N-n}\frac{1-f}{n}\rho C_xC_y\right)-$
$\bar{Y}^2\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2+\left[\frac{1-f}{n}+\frac{1-f_1}{n_1}\right]P^2C_x^2+2\frac{1-f}{n}P\rho C_xC_y\right)>0$

$$\left(\frac{1-f}{n}C_y^2+\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}+\frac{1-f_1}{n_1}\right)C_x^2-2\frac{n}{N-n}\frac{1-f}{n}\rho C_xC_y\right)$$
$$>\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2+\left[\frac{1-f}{n}+\frac{1-f_1}{n_1}\right]P^2C_x^2+2\frac{1-f}{n}P\rho C_xC_y\right)$$
$$\left(1-\left(\frac{n}{N-n}\right)^2\right)C_y^2+\left(\frac{n}{N-n}\right)^2\left((1+\delta)C_x^2-2\frac{n}{N-n}\rho C_yC_x\right)$$
$$>\left(\frac{n}{N-n}\right)^2\left((1+\delta)P^2C_x^2+2P\rho C_yC_x\right)$$
$$\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2}+\left((1+\delta)-2\frac{n}{N-n}v\right)>P\left((1+\delta)P+2v\right)$$
$$\left(\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2}+\left((1+\delta)-2v\left[1+\frac{n}{N-n}\right]\right)\right)/(1+\delta)<P$$
$$<\left(\frac{N(N-2n)}{n^2}\right)\frac{C_y^2}{C_x^2}+\left((1+\delta)-2\frac{n}{N-n}v\right)$$

Which is positive if $0<n<\frac{N}{2}$ and $\left(1-\frac{4v}{(1+\delta)}\right)<P<\left((1+\delta)-2v\right)$

(iv) $MSE(\bar{y}_{FTD})_{II}-MSE(\bar{y}_{FTD}^{+})_{II}=\bar{Y}^2\left(\frac{1-f}{n}C_y^2+\left(\frac{1-f}{n}+\frac{1-f_1}{n_1}\right)P^2C_x^2+2\frac{1-f}{n}P\rho C_xC_y\right)-$
$\bar{Y}^2\left(\frac{n}{N-n}\right)^2\left(\frac{1-f}{n}C_y^2+\left[\frac{1-f}{n}+\frac{1-f_1}{n_1}\right]P^2C_x^2+2\frac{1-f}{n}P\rho C_xC_y\right)>0$

$$1>\left(\frac{n}{N-n}\right)^2$$

Which is positive if $0<n<\frac{N}{2}$

Empirical Study

In order to demonstrate numerically the efficiency of the proposed estimator, we consider the data below

Population I: [Source: (Das, 1988)]

X: The number of agricultural laborers 1961

Y: The number of agricultural laborers 1971

$\bar{Y}=39.0680,\quad \bar{X}=25.1110,\quad N=278,\quad n=60,\quad n_1=180,$

$C_y=1.4451,\quad C_x=1.6198,\quad \rho_{yx}=0.7213$

Population II: [Source: (Cochran, 1977)]
X: The number of rooms per block
Y: The number of persons per block
$\bar{Y} = 101.1, \quad \bar{X} = 58.80, \quad N = 20, \quad n = 8, \quad n_1 = 12,$
$C_y = 0.14450, \quad C_x = 0.1281, \quad \rho_{yx} = 0.650$

Table 1: Optimum values of d

Optimum value of d	Population I		Population II	
	Case I (17)	Case II (22)	Case I (17)	Case II (22)
d_1	7.5191	10.0356	6.6414	6.9196
d_2	1.5283	1.3363	1.5898	1.4924
d_3	2.7570	2.9745	2.8030	3.0581
$P_{optimum}$	-0.6435	-0.7332	0.5596	-0.5076

Table 1 show the roots of the cubic equations (17) and (22) which optimize the value of P and thereby minimizes the MSE of the proposed estimator for the populations considered in the study.

Table 2:
Percentage relative efficiencies of different estimators with respect to $\bar{y}$

Estimator	Population I		Population II	
	Case I	Case II	Case I	Case II
$\bar{y}$	100.00	100.00	100.00	100.00
$\bar{y}_r$	144.21	120.82	125.57	101.76
$\bar{y}_{st}$	142.34	150.51	130.34	135.83
$\bar{y}_{FT}$	179.31	182.64	129.19	141.17
$\bar{y}^*_{FT}$	2367.1	2411.0	290.67	317.64

Conclusion

Table 1 shows the optimum values of d for which the proposed estimator attains optimality in both cases I and II under the two populations considered in the study. Table 2 also depicts the efficiencies of the proposed estimator and traditional estimators and it is observed that the percent relative efficiency of the proposed estimator with respect to that of sample mean is higher and superior over the percent relative efficiencies of ratio estimator, Strivenkataramana estimator and factor-type estimator. It is also observed that all the estimators considered in the study performed better under case II than case I with exception of classical ratio estimator. This implies that if second sample is drawn as an independent sample from population i.e. $S \subset \Omega$ though cost of sampling might be increased, there would be increase in the efficiency and accuracy of the estimate due more information about population are obtained.

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