ON CONTROLLABILITY PROPERTY OF OPTIMAL CONTROL MODEL OF ELECTRIC POWER GENERATING SYSTEM

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Abstract
Electricity is the key energy source for industrial, commercial and domestic activities in the modern world. Bamigbola and Aderinto(2009). Mathematical control deals with the basic principles underlying the analysis and design of a system, to control an object means to influence its behavior so as to achieve a desired goal. In this paper, we present controllability properties of electric power generating system model in an attempt to have better understanding of the study of electric power generating system.

Keywords: Mathematical model, Controllability, Electricity Generation, Optimal control

Introduction
The Nigerian power system is plagued with many problems due to insufficient generation, unreliable transmission and distribution lines, uncontrolled and unknown load distributions, in addition to system dynamic constraints imposed by the longitudinal nature of the whole system. In view of these constraints, the load demands are never met leading to poor voltage regulation, Manafa (1978), Olle(1987). Hence, there is need for increasing production level, higher reliability, and controllability level to attain optimal level.

The concept of controllability was introduced by Kalman in 1960 as a fundamental to modern control theory. If a control function $u(t)$ can be found in a system which will transform the initial state x_0 of a system to some desired final state x_f in finite time, then the system is controllable, Burghes et al.(1989).

A state system is said to be controllable if a control function $u(t)$ can be found which will transform the initial state x_0 of a system to some desired fund state x in finite time. A large body of literature exists for operational design of the optimal electric power flow, among are, Aderinto and Bamigbola (2010), Bouktir and Sliman (2005), Eti et al. (2004), Komolafe et al. (2009), Dmytro et al. (2007) and George et al. (1996). Hence, the goal of this paper is to study the controllability property of electric power generating system model for better understanding of the system.

Mathematical Model

Consider the electric power generating system model, by Aderinto and Bamigbola(2010)

$$\frac{dG_i(t)}{dt} = \alpha_i + q_i C_i(t) G_i(t) - k_i G_i(t), \qquad i = 1, 2, \dots, m \qquad (1)$$

$$\frac{dC_i(t)}{dt} = (s_i + y_i) + r_i C_i(t) G_i(t) - x_i u_i(t) C_i(t) + \gamma_i C_i(t), \quad i = 1, 2, ..., m \qquad (2)$$

Where, $G_i(t)$ represents the amount of power generated by the i^th generator at time t, $C_i(t)$ the capital investment on the i^{th} generator at time t. $\alpha_i, q_{i,}, \ and \ k_i$ are respectively the actual mechanical/electrical energy from the high pressure turbine and low pressure turbine , the corresponding running cost and the transmission loss rate, which depends on the distance from the grid center. Also, the capital $C_i(t)$ *at* time t is known to be dependent on labour cost s_i , maintenance y_i , fuel cost $r_i C_i(t) G_i(t)$, capacity rate x_i and the .cost of transmission to the grid centre $\gamma_i C(t)$, $u_i(t)$, $a_i \le u_i \le b_i$ $(i = 1, ..., m)$ the generator capacity as our control since we cannot neglect the operation limitation on the equipment because of its lifespan, the upper bound for $u_i(t)$ is choosing to be 1, to represent the total capability of the machine.

In the above setting an important objective is to minimize the total operating cost incurred in the process of generating the required quantity of electric power *G* at any time t, thus, the expression for the objective function is of the form

$$J(u) = \int_{t_0}^{t_f} [\delta^T C(t) + \eta u^T u] \, \mathrm{dt} \qquad (3)$$

Where $\delta = (\delta_1, \delta_2, ..., \delta_m)^T$ is the unit of power generating, η (a parameter to balance the size of the control) is the weight on the benefit and cost. The benefit based on the power output being maximized by the capacity of the generating machine and the percentage effect of the cost being minimized. Hence, if *u (t)* =1 represent maximal capacity of the generating machine, then the maximal cost is represented by u^2.

Let

$$G_i = (G_1, G_2, ..., G_m)^T$$

$$C = (C_1, C_2, ..., C_m)^T$$

$$\alpha_i = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m)^T$$

$$u_i = (u_1, u_2, ..., u_m)^T$$

$$k_i = (k_1, k_2, ..., k_m)^T$$

$$\gamma_i = (\gamma_1, \gamma_2, \gamma_3, ..., \gamma_m)^T$$

$$(s_i + y_i) = (s_1 + y_1, .., s_m + y_m)^T$$

$$A = \begin{pmatrix} q_1 & 0 & . & . & 0 \\ 0 & q_2 & 0 & . & 0 \\ 0 & 0 & q_3 & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & q_m \end{pmatrix}, \quad D = \begin{pmatrix} r_1 & 0 & . & . & 0 \\ 0 & r_2 & 0 & . & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & r_m \end{pmatrix}, \quad \text{and } E = \begin{pmatrix} x_1 & 0 & . & . & 0 \\ 0 & x_2 & 0 & . & 0 \\ 0 & 0 & x_3 & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & x_m \end{pmatrix}$$

Equations (1) and (2) can be rewritten in the following matrix- vector form as:

$$\frac{dG(t)}{dt} = \alpha + C(t)AG(t) - kG(t) \quad (4)$$

and

$$\frac{dC(t)}{dt} = (s+y) + C(t)DG(t) - u(t)EC(t) + \gamma_i C(t) \quad (5)$$

$$G(t_0) = G_0, C(t_0) = C_0, \quad a_i \le u_i \le b_i \quad (6)$$

Controllability Properties

Controllability of a system was introduced by Kalman in 1960 as a fundamental to modern control theory, if and only if a control function $u(t)$ can be found in a system which will transform the initial state x_0 to a specified final state x_f, then the system is controllable. For complete state controllability, each of the components of x must be transformed by an appropriate $u(t)$. It also follows that for a controllable system the transformed matrix must be non-singular, Burghes et al.(1989). In this section we present the controllability properties of electric power generating system model for better understanding of the system.

Definition 3.1

Consider the system of equation

$$\frac{d\bar{x}}{dt} = A(t)\bar{x}(t) + B(t)\bar{u}(t) \quad (7)$$

$\bar{x}(t) = \bar{x}_0, A(t)$ *is* $n \times n$, $B(t)$ *is* $n \times m$ *matrices*,

$\bar{u}(t) \in R^m$ *and* $\bar{x}(t) \in R^n, u(t)$ is called the control $\bar{x}(t)$ is the corresponding state system.

The linear system (7) is said to be controllable if given any initial state $\bar{x}_0$ and any final state $\bar{x}_f$ in R^n, then there exist a control $\bar{u}(t)$ so that the corresponding state $\bar{x}(t)$ of equation (7) satisfies the condition, $\bar{x}(t_0) = \bar{x}_0,\ \bar{x}(t_f) = \bar{x}_f$,

The control $\bar{u}(t)$ is said to steer the state from the initial state $\bar{x}_0$ to the final state $\bar{x}_f$.

Let $\phi(t, t_0)$ be the transition matrix of equation (7), then by the variation of parameter formula it can be written as

$$\bar{x}(t) = \phi(t, t_0)\bar{x}_0 + \int_{t_0}^{t} \phi(t,\tau)B(\tau)\bar{u}(\tau)d\tau \quad (7a)$$

As $(t,\tau) \to \phi(t,\tau)$ is continuous, it follows that $\|\phi(t,\tau)\| \le M \ \forall\, t,\tau \in [t_0, t_f]$

Hence, controllability of the system in equation (7) is the same as to finding

$$\bar{u}(t) : \bar{x}_f = \bar{x}(t_f) = \phi(t_f, t_0)\bar{x}_0 + \int_{t_0}^{t_f} \phi(t_f,\tau)B(\tau)\bar{u}(\tau)d\tau$$

i.e $\quad \bar{x}(t_f) - \phi\left(t_f, t_0\right)\bar{x}_0 = \int_{t_0}^{t_f} \phi(t_f, \tau) B(\tau)\bar{u}(\tau) d\tau \qquad (7b)$

If we define linear operator $\mathrm{L} : \mathrm{X} = \mathrm{L}_2([t_0, \mathrm{t}_f]\square^n)$ by

$$[\mathrm{L}\bar{\mathrm{u}}] = \int_{t_0}^{t_f} \phi(t_f, \tau) B(\tau)\bar{u}(\tau) d\tau \qquad (7c)$$

Then by (7b), the controllability property is equivalent to showing that operator L is surjective. If $\mathrm{L} : (\mathrm{X}, \mathrm{Y})$ then $N(L)$ and $R(L)$ are defined as $N(L) = \{\bar{x} \in X : L\bar{x} = 0\}, R(L) = \{\bar{y} \in Y : \bar{y} = L\bar{x}, \bar{x} \in X\}$

Lemma 3.1: Mohan (2006)
The system given by equation (7) is controllable if and only if

$$W\left(t_0, t_f\right) = \int_{t_0}^{t_f} \left[\varphi\left(t_f, \tau\right) B(\tau) B^T(\tau) \varphi^T\left(t, t_f\right)\right] d\tau \qquad (8)$$

is nonsingular, where $W\left(t_0, t_f\right)$ is known as the controllability Grammian

A control $\bar{u}(t)$ steering the system from the initial state $\bar{x}_0$ to the final $\bar{x}_f$ is given by

$$\bar{u}(t) = B^T(t)\varphi^T\left(t_f, t\right)\left[W\left(t_0, t_f\right)\right]^{-1}\left[x_f - \varphi\left(t_f, t_0\right)x_0\right] \qquad (9)$$

Lemma 3.2: Mohan (2006)
The system of equation (7) is controllable if
(i) $W\left(t_0, t_f\right)$ is symmetric and positive semi definite.
(ii) $W\left(t_0, t_f\right)$ satisfies the linear differential equation

$$\frac{d}{dt}\left[W\left(t_0, t_f\right)\right] = A(t)W\left(t_0, t\right) + W\left(t_f, t\right)W\left(t, t_f\right)A^T(t) + B(t)B^T(t) \qquad (10)$$

$$W\left(t_0, t_0\right) = 0$$

(iii) $W\left(t_0, t_f\right)$ satisfies functional equation

$$W\left(t_0, t_f\right) = W\left(t_0, t\right) + \phi\left(t_f, t\right)W\left(t, t_f\right)\phi^T\left(t_f, t\right) \qquad (11)$$

Proof: see Mohan (2006)

Theorem 3.1

The system

$$\frac{dG(t)}{dt} = \alpha + C(t)AG(t) - kG(t)$$

and

$$\frac{dC(t)}{dt} = (s + y) + C(t)DG(t) - u(t)EC(t) + \gamma_i C(t)$$

at the initial time

$G(t_0)=G_0,\quad C(t_0)=C_0,\ a_i \le u_i \le b_i$

given by equations (4), (5), and (6) above, is controllable if and only if

$$W_1(t_0,t_f)=\int_{t_0}^{t_f}\left[\varphi(t_f,\tau)E(\tau)E^T(\tau)\varphi^T(t,t_f)\right]d\tau \qquad (12)$$

is nonsingular.

Proof:

Using Lemmas 1 and 2, assuming that $\varphi(t_f,\tau)$ is the transition matrix of the above system and by the variation of parameter in (7a), equation (6) is equivalent to

$$C(t)=\phi(t,t_0)G_0+\int_{t_0}^{t}\phi(t_f,\tau)E(\tau)\bar{u}(\tau)d\tau \qquad (13)$$

As $(t,\tau)\to\phi(t,t_0)$ is continuous, it follows that

$\|\phi(t_f,\tau)\|\le M\ \forall\ t,\tau\ \in(t_0,t_f)$

Hence, controllability of (4 - 6) is equivalent to finding $\bar{u}(t)$ such that

$$C_f=C(t_f)=\varphi(t_f,t_0)+\int_{t_0}^{t_f}\varphi(t_f,\tau)E(\tau)u(\tau)d\tau$$

$$i.e,\ C(t_f)-\phi(t_f,t_0)C_0=\int_{t_0}^{t_f}\phi(t_f,\tau)E(\tau)\bar{u}(\tau)d\tau \qquad (14)$$

Let $L:X=C(t)=L_2([t_0,t_f],R^m)$ where L is a linear operator defined by

$$|L\bar{u}|=\int_{t_0}^{t_f}\left[\phi(t_f,\tau)E(\tau)u(\tau)\right]d\tau \qquad (15)$$

In view of equation (15) controllability property reduced to showing that operator L is surjective.

Let (C_0,C_f) *be* any pair of vector in $R^n\times R^n$. We are to determine $u\in C_i$ such that

$$\int_{t_0}^{t_f}\varphi(t_f,\tau)E(\tau)u(\tau)d\tau=C_z=C_f-(\varphi(t_f,t_0))C_0$$

i.e solving $|L\bar{u}|=zf$ in the space X for a given $\bar{z}f$

Let L be a bounded linear operator as $\varphi(t,\tau)$ is bounded. Also, let $L^*:R^n\to X$ be its adjoint operator, for $u\in X$ and $\alpha\in R^n$, we have

$$(L\bar{u},\alpha)R^n=\left[\bar{\alpha},\int_{t_0}^{t_f}\varphi(t_f,\tau)E(\tau)\bar{u}(\tau)d\tau\right]R^n$$

$$=\int_{t_0}^{t_f}\alpha,\varphi(t_{f,}\tau)E(\tau)u(\tau)R^n d\tau$$

$$= \int_{t_0}^{t_f} \left(E^T(\tau)\varphi^T\left(t_f,\tau\right)\alpha\ u(\tau)\right)R^n d\tau \qquad = \left(\bar{u}, L^*\alpha\right)_X$$

$$\Rightarrow \left[L^{*\cdot}\bar{\alpha}\right](t) = E^T(t)\varphi^T\left(t_f,t\right)\alpha \qquad (16)$$

To show that L is onto, we show that

$LL^*: R^n \to R^n$ (nonsingular) i.e , given any $z_f \in R^n$, *find* $\alpha \in R^n$ such that

$$LL^*_\alpha : z_f \qquad (17)$$

If equation (17) is solvable, then a control u which solves $Lu = zf$ *is* $\bar{u} = L^*\alpha$

Computing LL^*, we have

$$\left[LL^*\right](\alpha) = L\left(E^T(t)\varphi^T\left(t_f,t\right)\alpha\right)$$

$$= \left[\int_{t_0}^{t_f}\left[\varphi\left(t_f,t\right)E(t)E^T(t)\varphi^T\left(t_f,t\right)dt\right]\right]\alpha$$

$$= W\left(t_0,t_f\right)\alpha$$

If $W\left(t_0,t_f\right)$ is nonsingular, we have a control u(t) given by

$$u(t) = L^*\alpha = L^*\left(LL^*\right)^{-1} zf = L^*\left(W\left(t_0,t_f\right)\right)^{-1}$$

$$= E^T(t)\varphi^T\left(t_f,t_0\right)\left[W\left(t_0,t_f\right)\right]^{-1}\left[C_f - z_f\left(t_f,t_0\right)G_0\right]$$

This will steer the state from the initial to the final state. Conversely, let equation (4-6) be controllable, i.e. L is onto

i.e ., $R(L) = R^n$, *then* $N\left(L^*\right) = \left[R(L)\right]^{-1} = \left[R^0\right]^{-1} = \{0\}$.

hence L^* *is* $1-1 \Rightarrow LL^*$ is also 1-1, which is proved as below

Assume that $LL^*\alpha = 0 \Rightarrow 0 = \left(LL^*\alpha,\alpha\right) = \left\|L^*\alpha\right\|^2$ and hence

$L^*\alpha = 0$, $\quad L^*1-1 \Rightarrow \alpha = 0.$ *Thus* $N\left(LL^*\right) = \{0\}$

Thus $W\left(t_0,t_f\right) = LL^*: R^n \to R^n$ is 1-1

and hence nonsingular.

Theorem 3.2

The system of equation given by equation (4-6) is controllable if (i) $W\left(t_0,t_f\right)$ is symmetric and positive semi definite.

(ii) $W\left(t_0,t_f\right)$ satisfies $\dfrac{d}{dt}\left(W\left(t_0,t\right)\right) = G(t)DW\left(t_0,t\right) + W\left(t_0,t\right)D^T + EE^TW\left(t_0,t_f\right), W\left(t_0,t_0\right) = 0$

(iii) $W\left(t_0,t_f\right) = W\left(t_0,t\right) + \varphi\left(t_f,t\right)W\left(t,t_f\right)\varphi^T\left(t_f,t\right)$

Proof:

Let $W\left(t_0,t_f\right) = \displaystyle\int_{t_0}^{t_f}\left[\varphi\left(t_f,\tau\right)E(\tau)E^T(\tau)\varphi^T\left(t_f,\tau\right)\right]d\tau$

Since $W^T\left(t_0,t_f\right) = W\left(t_0,t_f\right)$, *then* $W\left(t_0,t_f\right)$ is symmetric. Hence,

$$\left(W(t_0,t_f)\alpha,\,\alpha\right)_{R^n} = \int_{t_0}^{,t_f}\left[\varphi(t_f,\tau)\,E(\tau)E^T(\tau)\varphi^T(t_f,\tau)\overline{\alpha},\overline{\alpha}\right]_{R^n} d\tau$$

$$= \int_{t_0}^{t_f}\left\|E^T(\tau)\varphi^T(t_f,\tau)\overline{\alpha}\right\|^2$$

$$\geq \quad 0 \ \ \forall \ \ \overline{\alpha} \in R^n$$

(ii) Using Leibnitz rule, we have

$$\frac{d}{dt}(W(t_0,t)) = \frac{d}{dt}\left[\int_{t_0}^{t}\varphi(t_f,\tau)\,E(\tau)E^T(\tau)\varphi^T(t_f,\tau)d\tau\right]$$

$$= E(t)E^T(t) + \left[\int_{t_0}^{t}D(t)\,\varphi(t_f,\tau)\,E(\tau)E^T(\tau)\varphi^T(t_f,t)d\tau\right]$$

$$= E(t)E^T(t) + \left[\int_{t_0}^{t}D(t)\phi(t_f,\tau)\,E(\tau)E^T(\tau)\phi(t_f,t)d\tau\right]$$

$$+\left[\int_{t_0}^{t}\phi(t_f,\tau)\,E(\tau)E^T(\tau)\phi^T(t_f,t)D^T(t)\,d\tau\right]$$

$$= D(t)W(t_0,t) + W(t_0,t)D^T(t) + E(t)E^T(t)$$

$$(iii) \quad W(t_0,t) = \int_{t_0}^{t}\varphi(t_f,t)E(\tau)E^T\varphi^T(t_f,t)d\tau + \int_{t}^{t_f}(t_f,\tau)E(\tau)E^T\varphi(t_f,t)d\tau$$

$$= W(t_0,t) + \varphi(t_f,t)\left(\int_{t}^{t_f}\varphi(t,\tau)E(\tau)E^T(\tau)\varphi^T(t,\tau)d\tau)\varphi^T(t_{f,}t)\right)$$

$$= W(t_0,t) + \varphi(t_f,t)W(t,t_f)\varphi^T(t_{f,}t)$$

Hence, the system given by equations (4), (5), and (6) is controllable.

The system can then be solved using appropriate numerical method, specifically, Runge-kutta fourth- order method because of its higher level of accuracy than the first, second and third orders, Gerald and Wheatly (1994), Jain (1983), Pingping(2009), Poppe and Kautz (1998), Luke(1982), Steven(2007).

The optimal control model is solved using real life data obtained from National Control Centre, Osogbo, Aderinto and Bamigbola (2012). However, the result obtained shows that control is very important in order to generate more with minimum cost and still maintain good condition of the generating machine, Aderinto and Bamigbola (2012).

Conclusion

The availability of electric power, in the right quantity and quality, is essential to the technological and industrial advancement in today's world. The application of optimal control theory to the mathematical model of the electric power generating system was to improve the quality and quantity of electric power generated and minimizes the cost of production under secured condition. In this work, the controllability properties of the optimal control model were investigated and the system was found controllable. And since controllability is one of the characteristics that guaranteed uniqueness of solution, hence our result is unque.

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