# CONSTRUCTION AND IMPLEMENTATION OF SOME HYBRID SCHEMES FOR SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (ODEs)

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## Abstract
*In this research work, we are concerned with the construction and implementation of block hybrid methods for solving ordinary differential equations using power series. We equally develop a block hybrid method of step three (i.e. k = 3) with two off-grid points at collation. We also analysed the order, error constant and convergence properties of the schemes derived. A numerical experiment of the block method was done on both stiff and non-stiff problems and the results showed a high degree of consistency with the theoretical nalysis.*

Keywords: Collocation, Off-grid Points, Linear Multi-step Methods (LMMs), Continuous Schemes, Zero-stability, Initial Value Problems (IVP), Stiff and Non-stiff Problems

## Introduction
The introduction of the continuous collocation method has been able to bridge the gap between the discrete collocation methods and the conventional Multistep methods. Thus, it is possible to write the Linear Multistep Methods (LMMs) in the form of some continuous collocation schemes. (Okunuga & Ehigie, 2009).

Onumanyi, Oladele, Adeniyi and Awoyemi (1993), Adeniyi and Alabi (2006) as well as Fatokun (2007) have all introduced other variants of the collocation method, which recently led to some continuous collocation approach. The introduction of the continuous collocation schemes as against the discrete schemes includes the fact that better global error can be estimated and approximations can be equally obtained at all interior points.

## Methodology
Consider the initial value problem;

$$y' = f(x,y); \quad a \le x \le b, \quad y(a) = y_0 \tag{1}$$

On a given mesh $a = x_0 < x_1 < \cdots < x_n \ldots < X_m = b$

Where; $h = x_{n+1} - x_n, n = 0,1, \ldots\ldots\ldots, N$ where $h$ is considered a constant step and $k$ is the step number of the method.

In order to solve the equation, Onumanyi *et al.* (1994) developed a linear multistep method with continuous coefficients by the idea of Multistep collocation.

Let; $$y(x) = \sum_{j=0}^{k+1} \alpha_j(x) y(x_{n+j}) + h \sum_{j=0}^{m-1} \beta(x) f\left(\bar{x}_j, \bar{y}(\bar{x}_j)\right) + \beta_v f(x_v, y_v) \tag{2}$$

Where; $$\alpha_j(x) = \sum_{j=0}^{k+m-1} \alpha_{j,i+1} x^i$$

$$h\beta_j(x) = \sum_{j=0}^{t+m-1} h\beta_{j,i+1}x^i \tag{3}$$

Also;

$$y(x_{n+j}) = y_{n+j}, j \in (0,1, \dots, k-1)$$
$$y'(\bar{x}_j) = f[x_j, y(\bar{x}_j)], j = 1, \dots, m \tag{4}$$

Satisfying the polynomial;

$$C^T\emptyset = \begin{pmatrix} \sum_{j=1}^{p} C_{ji\emptyset_j} \\ \vdots \\ \sum_{j=1}^{p} C_{jr\emptyset_j} \\ \sum_{j=1}^{p} C_{jr+1\emptyset_j} \\ \vdots \\ \sum_{j=1}^{p} C_{jp\emptyset_j} \end{pmatrix} \equiv \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \\ \beta_1 \\ \vdots \\ \beta_s \end{pmatrix} \tag{5}$$

To get $\alpha_j(x)$ and $\beta_j(x)$, Sirisena $(2003)$ arrived at the matrix equation of the form;

$$DC = I \tag{6}$$

Where $I$ is the identity matrix of dimension$(t+m) \times (t+m)$. $D$ and $C$ are defined as;

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & . & . & . & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & . & . & . & x_{n+1}^{t+m-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & . & . & . & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & . & . & . & (t+m-1)x_0^{t+m-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 1 & 2\bar{x}_{m-1} & . & . & . & (t+m-1)\bar{x}_{m-1}^{t+m-2} \end{pmatrix} \tag{7}$$

The matrix $(7)$ is the multistep collocation matrix of dimension $(t+m) \times (t+m)$; whose matrix $C$ has the same dimension and whose columns give the continuous coefficients as;

$$C = \begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & . & . & . & \alpha_{t-1,1} & h\beta_{0,1} & . & . & . & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & . & . & . & \alpha_{t-1,2} & h\beta_{0,2} & . & . & . & h\beta_{m-1,2} \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ \alpha_{0,t+m} & \alpha_{1,t+m} & . & . & . & \alpha_{t-1,t+m} & h\beta_{0,t+m} & . & . & . & h\beta_{m-1,t+m} \end{pmatrix} \tag{8}$$

We defined $t$ as the number of interpolation points and $m$ as the number of collocation points used. And from $(6)$ we noticed that; $C = D^{-1}$.

### Derivation of the Block Hybrid with Two Off-grid Points for $k = 3$;

The general form of the block method from (2) with two off-grid points at collocation for $k = 3$ will be;

$$y(x) = \alpha_0 y_{n+1} + h[\beta_0 f_{n+1} + \beta_1 f_{n+2} + \beta_2 f_{n+3} + \beta_3 f_{n+c} + \beta_4 f_{n+d}] \quad (9)$$

The matrix of the proposed method with two off-grid points at collocation for $k = 3$ is designed as;

$$\begin{pmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 \\ 0 & 1 & 2x_{n+c} & 3x_{n+c}^2 & 4x_{n+c}^3 & 5x_{n+c}^4 \\ 0 & 1 & 2x_{n+d} & 3x_{n+d}^2 & 4x_{n+d}^3 & 5x_{n+d}^4 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+c} \\ f_{n+d} \end{pmatrix} \quad (10)$$

Solving the above matrix in equation (10), by using any of the different methods of solving matrix method with the help of maple software, which when evaluated at $x = x[n] + 3 * h$, $x = x[n] + \frac{5}{2} * h$, $x = x[n] + 2 * h$, $x = x[n] + \frac{3}{2} * h$, $x = x[n] + \frac{1}{2} * h$ and $x = x[n]$ and where $c = 1/2 \ when \ d = 5/2$, we are able to derive the following schemes;

$$y_{n+3} = y_{n+1} + \frac{23}{45} hf_{n+1} + \frac{44}{45} hf_{n+2} + \frac{17}{75} hf_{n+3} - \frac{16}{225} hf_{n+\frac{1}{2}} + \frac{16}{45} hf_{n+\frac{5}{2}}$$

$$y_{n+\frac{5}{2}} = y_{n+1} + \frac{39}{80} hf_{n+1} + \frac{87}{80} hf_{n+2} + \frac{9}{200} hf_{n+3} - \frac{51}{800} hf_{n+\frac{1}{2}} - \frac{9}{160} hf_{n+\frac{5}{2}}$$

$$y_{n+2} = y_{n+1} + \frac{91}{180} hf_{n+1} + \frac{37}{45} hf_{n+2} + \frac{19}{300} hf_{n+3} - \frac{31}{450} hf_{n+\frac{1}{2}} - \frac{29}{90} hf_{n+\frac{5}{2}}$$

$$y_{n+\frac{3}{2}} = y_{n+1} + \frac{287}{720} hf_{n+1} + \frac{191}{720} hf_{n+2} + \frac{19}{600} hf_{n+3} - \frac{323}{7200} hf_{n+\frac{1}{2}} - \frac{217}{1440} hf_{n+\frac{5}{2}}$$

$$y_{n+\frac{1}{2}} = y_{n+1} - \frac{257}{720} hf_{n+1} + \frac{79}{720} hf_{n+2} + \frac{11}{600} hf_{n+3} - \frac{1387}{7200} hf_{n+\frac{1}{2}} - \frac{113}{1440} hf_{n+\frac{5}{2}}$$

$$y_n = y_{n+1} + \frac{119}{180} hf_{n+1} - \frac{37}{45} hf_{n+2} - \frac{49}{300} hf_{n+3} - \frac{599}{450} hf_{n+\frac{1}{2}} + \frac{59}{90} hf_{n+\frac{5}{2}} \quad (11)$$

The above equation (11) shows that the schemes are zero-stable of order $(5,5,5,5,5,5)^T$ with their error constants $C_6 = \left[-\frac{1}{450}, -\frac{87}{51200}, -\frac{19}{9600}, -\frac{511}{460800}, -\frac{133}{153600}, \frac{287}{28800}\right]^T$ respectively.

## Convergence Analysis

The Fatunla (1991) approach states that, the block method can be presented as a single block r-point multistep method of the form;

$$A^{(0)}Y_m = \sum_{i=1}^{k} A^{(i)}Y_{m-1} + h\sum_{i=0}^{k} B^{(i)}F_{m-1} \qquad (12)$$

The schemes of the block hybrid method in equation (11) above for $k = 3$ are put in matrix form as $A_0$, $A_1$, $B_0 and B_1$. We also have to multiply the matrix vectors with the inverse of $A_0$ i.e. $[A_0]^{-1}$ to obtain the normalized form $A^{(0)}$, following Fatunla (1992, 1994) as:

$$A^{(0)} = {A_0}^{-1}A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (13)$$

The above block method is the normalized form of the above schemes for two off-grid points at collocation of order $k = 3$, according to Yahaya (2004) and Umar (2009).

The characteristic polynomial of the above block method is;

$$\rho(\lambda) = \mathbf{det}[\lambda A^{(0)} - A^{(1)}]$$

$$\rho(\lambda) = \left| \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right|$$

$$\rho(\lambda) = \begin{vmatrix} \lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 1 \end{vmatrix} \qquad (14)$$

Solving the above determinant yields;

$$\rho(\lambda) = \lambda^5(\lambda - 1) = 0 \qquad (15)$$

This implies that;

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0, \text{ while; } \lambda_6 = 1. \qquad (16)$$

This implies zero-stability. The above block method is also consistent, since it is of order $\rho > 1$, as stated by Fatunla (1994). Thus, the above block method is convergent.

## Numerical Experiments

To illustrate the performance of our proposed schemes, we will compare the performance of our schemes with the exact solution and the results obtained from Adegboye (2007).

### Example 5.1: [CF: Adegboye (2007)]

Consider the initial value problem,

$y' = -y, \qquad y(0) = 1$

$0 \le x \le 2 \qquad □ = 0.1$

*exact solution:* $y(x) = e^{-x}$ (17)

We have to transform the schemes by substitution, in order to get a recurrence relation. Substituting for $k = 0, 3, 6, 9, \ldots\ldots\ldots$ and solving simultaneously, we obtained the required result displayed in the table below.

Table 1: Example of IVP 5.1, for k= 3 with two off-grid points at Collocation. See (11)

| N | X | Exact Solution | Numerical Solution (Derived) | Error (Derived) | Numerical Solution (Adegboye) | Error Adegboye |
|---|---|---|---|---|---|---|
| 0 | 0 | 1.000000000 | 1.000000000 | 0 | 1.000000000 | 0.00E+00 |
| 1 | 0.1 | 0.904837418 | 0.904837425 | 7.00E-09 | 0.904837328 | 9.04E-08 |
| 2 | 0.2 | 0.818730753 | 0.818730761 | 8.00E-09 | 0.818730668 | 8.56E-08 |
| 3 | 0.3 | 0.740818221 | 0.740818228 | 7.00E-09 | 0.740818069 | 1.52E-07 |
| 4 | 0.4 | 0.670320046 | 0.670320058 | 1.20E-08 | 0.670319906 | 1.40E-07 |
| 5 | 0.5 | 0.606530660 | 0.606530672 | 1.20E-08 | 0.606530472 | 1.88E-07 |
| 6 | 0.6 | 0.548811636 | 0.548811647 | 1.10E-08 | 0.548811464 | 1.72E-07 |
| 7 | 0.7 | 0.496585304 | 0.496585318 | 1.40E-08 | 0.496585098 | 2.06E-07 |
| 8 | 0.8 | 0.449328964 | 0.449328978 | 1.40E-08 | 0.449328777 | 1.88E-07 |
| 9 | 0.9 | 0.406569660 | 0.406569672 | 1.20E-08 | 0.406569449 | 2.10E-07 |
| 10 | 1.0 | 0.367879441 | 0.367879456 | 1.50E-08 | 0.367879249 | 1.92E-07 |

### Example 5.2: [CF: Adegboye (2007)]

Consider the initial value problem,

$□' = -9□, \qquad □(0) = □$

$0 \le □ \le 2 \qquad □ = 0.1$

□□□□□ □□□□□□□□: $□(□) = □^{1-9□}$ (18)

We again have to transform the schemes by substitution, in order to get a recurrence relation as above. Substituting for $\square = 0, 3, 6, 9, \ldots\ldots$ and solving simultaneously, we obtained the required result displayed in the table below.

Table 2: Example of IVP 5.2, for k=3 with two off-grid points at Collocation. See (11)

| N | X | Exact Solution | Numerical Solution (Derived) | Error (Derived) | Numerical Solution (Adegboye) | Error Adegboye |
|---|---|---|---|---|---|---|
| 0 | 0 | 2.718281828 | 2.718281828 | 0 | 2.718281828 | 0.00E+00 |
| 1 | 0.1 | 1.105170918 | 1.106518698 | 1.35E-03 | 1.097149097 | 8.02E-03 |
| 2 | 0.2 | 0.449328964 | 0.450267761 | 9.39E-04 | 0.450937202 | 1.61E-03 |
| 3 | 0.3 | 0.182683524 | 0.183155068 | 4.72E-04 | 0.174522866 | 8.16E-03 |
| 4 | 0.4 | 0.074273578 | 0.074556106 | 2.83E-04 | 7.04E-02 | 3.83E-03 |
| 5 | 0.5 | 0.030197383 | 0.030338584 | 1.41E-04 | 2.90E-02 | 1.25E-03 |
| 6 | 0.6 | 0.012277340 | 0.012340802 | 6.35E-05 | 1.12E-02 | 1.07E-03 |
| 7 | 0.7 | 0.004991594 | 0.005023515 | 3.19E-05 | 4.52E-03 | 4.69E-04 |
| 8 | 0.8 | 0.002029431 | 0.002044183 | 1.48E-05 | 1.86E-03 | 1.71E-04 |
| 9 | 0.9 | 0.000825105 | 0.000831511 | 6.41E-06 | 7.19E-04 | 1.06E-04 |
| 10 | 1.0 | 0.000335463 | 0.000338479 | 3.02E-06 | 2.90E-04 | 4.51E-05 |

## Conclusion

From the tables of the results, we can conclude that, in solving both stiff and non-stiff problems, the derived hybrid schemes with two off-grid points at collocation yields better results than Adegboye (2007) hybrid schemes. Moreover, it can be discovered from the tables of results that, the results of numerical experiment with the block method approach when compared with that of exact solution yields an error that is significantly very minimal.

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