ON PARTICULAR SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS: A REVIEW

[1]M. H. ALI., [2]IBRAHIM YAHAYA MUHAMMAD., [2]MUHAMMAD MUDASSIR USMAN & [2]SUNUSI DAYYAB MUHAMMAD
[1]Department of Physics, Bayero University, Kano.
[2] Department of Physics, Kano University of Science and Technology, Wudil.
E-mail: alim@buk.edu.ng, Ibrahimyahaya29@yahoo.com, mmusman1007@gmail.com, sunusidayyab@gmail.com

Abstract
The particular solution of ordinary differential equations with constant coefficients is normally obtained using the method of undetermined coefficients where it is applicable. This method sometimes appears unnecessarily lengthy when inhomogeneous function contains product of trigonometric functions and polynomials. Strength and weakness of three alternatives to it are investigated using six examples. The result shows that the three methods performed better than the method of undetermined coefficients in all cases except when the inhomogeneous function is only exponential. Also the method in (Jia & Sogabe, 2013) excel in cases of higher order differential equations and lack of guessing in its procedure.

Keywords: Ordinary differential equation, Inhomogeneous equation

Introduction
The ubiquitousness of ordinary differential equations with constant coefficient in natural sciences cannot be over emphasized. The inhomogeneous differential equations with constant coefficient do exist also in many applications. The particular solution to the inhomogeneous part has two popular methods; the variation of parameters and the method of undetermined coefficients.

The method of undetermined coefficients is simpler compared to the general method of variation of parameters for lack of integration. However, besides its limitation to work with few functions, in some cases it turns out to be lengthy especially when the inhomogeneous function contains product of the appropriate functions. This leads to the evolution of additional alternatives, (De Leon, 2010; Gupta, 1998; Jia & Sogabe, 2013; Leon, 2015; Oliveira, 2012; Ortigueira, 2014).

Having more than one alternative means that each will have cases where it is better compared to the rest. In this work six examples are considered to investigate the strength and weakness of the three alternatives to the method of undetermined coefficients.

Procedure
The general procedure of obtaining solutions by four methods, are first presented as follow;

A. Method of Undetermined Coefficients: Rules of Thumb (Boyce & DiPrima, 1992) for ordinary differential equation of the form

$$ay'' + by' + cy = g(t), \qquad (1)$$

Where a, b and c are constants

(i) If an exponential function appears in $g(t)$, the guessing of the particular solution $Y(t)$ is an exponential function of the same exponent.

(ii) If a polynomial appears in $g(t)$, the guessing of $Y(t)$ is a generic polynomial of the same degree.

(iii) If $g(t)$ contains either cosine or sine functions, the guessing of $Y(t)$ have to contain both cosine and sine of the same frequency.

(iv) If $g(t)$ is a sum of several appropriate functions, separate it into n part and solve them individually (Superposition Methods).
(v) If $g(t)$ is a product of the normal functions, the starting choice for $Y(t)$ is chosen based on:
(a) $Y(t)$ is a product of the corresponding choices of all the parts of $g(t)$.
(b) There are as many coefficients as the number of distinct terms in $Y(t)$.
(c) Each distinct term must have its own coefficient, not shared with any other term.
i. Before finalizing the choice of $Y(t)$, compare it against the homogeneous solution $y(t)$. If there is any shared term between the two, the present choice of $Y(t)$ needs to be multiplied by t. Repeat until there is no shared term.

B. The Method of Gupta (Gupta, 1998)
The guessing for the particular solution is in general $Y(t) = u(t)e^{\alpha t}$ where u is a polynomial and α is a constant. The exponential term represents both the sine, cosine and the ordinary exponential functions. The polynomial $u(t)$ is obtained after substituting the guessed solution into the given inhomogeneous differential equation.

C. The De Oliveira Method (Oliveira, 2012)
The propose particular solution

$$Y(t) = Q(t)e^{\gamma t + i\delta} \quad (2)$$

Satisfying the differential equation

$$\frac{p^{n}(\gamma)}{n!}Q^{n} + \cdots + \frac{p'(\gamma)}{1!}Q' + \frac{p(\gamma)}{0!}Q = R \quad (3)$$

Moreover

a) If $\gamma \in \Re$, then we can suppose that Q is real. In such case, the real function $x(t) = Q(t)e^{\gamma t}$ is a solution of the initial equation.
b) If $\gamma \notin \Re$, then $z(t) = Q(t)e^{\gamma t + i\delta}$ is a complex solution. If $\gamma = \alpha + \beta i$, where $\alpha, \beta \in \Re$, then the function $x(t) = Re[z(t)]$ and $y(t) = Im[z(t)]$ satisfy
$P\left(\frac{d}{dt}\right)x = R(t)e^{\alpha t}\cos(\beta t + \delta)$, and $P\left(\frac{d}{dt}\right)y = R(t)e^{\alpha t}\sin(\beta t + \delta)$. (4)
c) If $p(\gamma) \neq 0$, then we have degree(Q) =degree(R).
d) If γ is a root of multiplicity k of the characteristic polynomial. Then we can choose a polynomial $Q(t) = t^{k}Q_{I}(t)$, with degree(Q_{I}) =degree(R)

D. The Jia and Sogabe Method (Jia & Sogabe, 2013)
This is similar to the Gupta's guess; $Y(t) = u(t)e^{\alpha t}$ but the $u(t)$ is directly calculated as follows;

(a) if is not the root of the characteristic equation of the homogeneous differential equation then

$$u(t) = \sum_{i=0}^{m} d_i g^{(i)}(t) \quad (5)$$

Where $d_0 = 1$ and $d_i = -\frac{k!}{p^{(k)}(\alpha)}\sum_{j=0}^{i-1}\frac{p^{(i-j)}(\alpha)}{(i-j)!}d_j$, for i=1, 2, 3...m also $g(t) = \frac{f(t)}{p^{(k)}(\alpha)}$

(b) if α is the root of the characteristic equation with multiplicity k, then

$$u^{(k)}(t) = \sum_{i=0}^{m} d_i g^{(i)}(t) \quad (6)$$

Where $d_0 = 1$ and $d_i = -\frac{k!}{p^{(k)}(\alpha)}\sum_{j=0}^{i-1}\frac{p^{(i+k-j)}(\alpha)}{(i+k-j)!}d_j$, for i=1,2,3..m also $g(t) = f(t)\cdot\frac{k!}{p^{(k)}(\alpha)}$

Each of the above four methods is used to solve the following six examples in order to assess the amount of the activities needed to arrive at a solution of the given example.

Example 1: Consider

$$y'' + y = tsint \quad (7)$$

The homogeneous solution is given as

$$y_h = c_1 sint + c_2 cost \quad (8)$$

Method of Undetermined Coefficient for Example 1

For the particular solution using the method of undetermined coefficient we guess

$$Y(t) = t(At + B)sint + t(Ct + D)cost \quad (9)$$

Since $sint$ is a solution to the homogeneous equation. This leads to the following derivatives;

$$Y'(t) = (At^2 + B)cost + (2At + B)sint - (Ct^2 + Dt)sint + (2Ct + D)cost \quad (10)$$

$$Y''(t) = (2At + B)cost - (At^2 + B)\sin t + (At + B)cost + (2A)sint - (Ct^2 + Dt)cos - (2Ct + D)s$$
$$(2C)cost - (2Ct + D)sint \quad (11)$$

Substituting (11) and (9) results into (7) gives

$A = -\frac{1}{4}$, $B = 0$, $C = 0$, and $D = -\frac{1}{4}$

This yield a particular solution

$$Y(t) = -\frac{1}{4}t^2 cost + \frac{t}{4}\sin t \quad (12)$$

Method of Gupta for Example 1

Based on this method we guess $Y(t)$ to be

$$Y(t) = u(t)e^{\alpha t} \quad (13)$$

Where $u(t)$ is a polynomial and differentiating $Y(t)$ we get

$$Y' = u'e^{\alpha t} + \alpha u e^{\alpha t} \quad (14)$$

$$Y'' = u''e^{\alpha t} + 2\alpha u'e^{\alpha t} + \alpha^2 u e^{\alpha t} \quad (15)$$

Substituting (13) and (15) in (7) gives

$$(u'' + 2\alpha u' + \alpha^2 u + u)e^{\alpha t} = te^{it} \quad (16)$$

Setting $\alpha = i$, and let $v = u'$ (16) becomes

$$v' + 2iv = t \quad (17)$$

Then we guess $v = At + B$, (17) finally gives

$A = -\frac{i}{4}$ and $B = \frac{1}{4}$

Therefore, $u(t) = \left(-\frac{i}{4} + \frac{1}{4}t\right)$ and choosing the imaginary part of our initial guessing of $Y(t)$ gives

$$Y(t) = -\frac{1}{4}t^2 cost + \frac{1}{4}t^2 sint \quad (18)$$

Method of De Oliveira for Example 1

This is similar to Gupta method in that, the sine function is replaced with exponential as

$$y'' + y = te^{it} \quad (19)$$

The characteristic polynomial is $p(\lambda) = \lambda^2 + 1$, $p'(\lambda) = 2\lambda$ and $p''(\lambda) = 2$. Inserting $p(\lambda)$ and its derivatives in the following equation

$$\left[\frac{p^{(n)}(\lambda)}{n!}q^{(n)} + \cdots + \frac{p''(\lambda)}{2!}q'' + \frac{p'(\lambda)}{1!}q' + p(\lambda)q\right]e^{\lambda t} = f(t) \quad (20)$$

Gives

$$[q'' + 2\lambda q' + (\lambda^2 + 1)q]e^{\lambda t} = te^{it} \quad (21)$$

with $\lambda = i$, (21) becomes

$$q'' + 2iq' = t \quad (22)$$

We guess $q = At^2 + Bt$, since $sint$ is solution to the homogeneous case. Differentiation of q gives $q' = 2At + B$ and $q'' = 2A$. Substituting q'' and q' in (22), solving for A and B gives

$A = -\frac{i}{4}$ and $B = \frac{1}{4}$

Then $q = -\frac{i}{4}t^2 + \frac{1}{4}t$ We have $y = qe^{\lambda t}$

$$Y(t) = \left(-\frac{i}{4}t^2 + \frac{1}{4}t\right)(cost + isint) \quad (23)$$

The particular solution is the imaginary part of y

$$Y(t) = -\frac{1}{4}t^2cost + \frac{1}{4}tsint \quad (24)$$

Method of Jia and Sogabe for Example 1

Changing the right hand side of (7) to complex form gives

$$y'' + y = te^{it} \quad (25)$$

Set $p(\lambda) = \lambda^2 + 1$, $f(t) = t$, $\alpha = i$. Since $p(\alpha) = 0$ and $p' \neq 0$, we have $k = 1$

Set $g(t) = f(t) \cdot \frac{k!}{p^{(k)}(\alpha)} = -\frac{1}{2}it$

Set $d_o = 1$ and $d_i = -\frac{k!}{p^{(k)}(\alpha)}\sum_{j=0}^{i-1}\frac{p^{(i+k-j)}(\alpha)}{(i+k-j)!}d_j$

By using d_i, we get

$$d_1 = -\frac{1}{p'(i)} \cdot \frac{p''(i)}{2!}d_o = -\frac{1}{2}i \quad (26)$$

Then set

$$u^{(k)}(t) = \sum_{i=0}^{m} d_i g^{(i)}(t)$$

$$u'(t) = \sum_{i=0}^{1} d_i g^{(i)}(t) = -\frac{1}{2}it + \frac{1}{4} \quad (27)$$

Hence

$$u(t) = -\frac{1}{2}it^2 + \frac{1}{4}t \quad (28)$$

Finally, we obtain the particular solution from the imaginary part of

$$y = u(t)e^{\alpha t} = \left(-\frac{i}{4}t^2 + \frac{1}{4}t\right)(cost + isint) \quad (29)$$

We have

$$y_p = -\frac{1}{4}t^2cost + \frac{1}{4}tsint \quad (30)$$

Example 2: Consider

$$y'' - 5y' + 6y = 2e^{3t} \quad (31)$$

The homogeneous solution is given as

$$y_h = c_1sint + c_2cost \quad (32)$$

Method of Undetermined Coefficient for Example 2

Our first choice is $Y(t) = Ce^{3t}$. But this is one of the solutions of the homogenous case. Therefore $Y(t) = Cte^{3t}$, $Y'(t) = 3Cte^{3t} + Ce^{3t}$ and $Y''(t) = 9Cte^{3t} + 3Ce^{3t} + 3Ce^{3t}$ putting $Y(t)$ and its derivatives in (29) gives

$$9Ce + 6Ce^{3t} - 15Cte^{3t} - 5Ce^{3t} + 6Cte^{3t} = 2e^{3t} \quad (33)$$

and $C = 2$ thus

$$Y(t) = 2te^{3t} \quad (34)$$

Method of Gupta for Example 2

Given $y'' - 5y' + 6y = 2e^{3t}$ (35)

Let $Y(t) = Ue^{\alpha t}$, $Y'(t) = U'e^{\alpha t} + U\alpha e^{\alpha t}$ and $Y''(t) = U''e^{\alpha t} + 2U'\alpha e^{\alpha t} + U\alpha^2e^{\alpha t}$

Substituting $Y(t)$ and its derivatives in (35) gives

$$U''e^{\alpha t} + 2U'\alpha e^{\alpha t} + U\alpha^2e^{\alpha t} - 5U'e^{\alpha t} - 5U\alpha e^{\alpha t} + 6Ue^{\alpha t} = 2e^{3t} \quad (36)$$

This shows that $\alpha = 0$ Therefore (36) becomes

$$U'' - 5U' + 6U = 2e^{3t} \quad (37)$$

Guessing $U = Ae^{3t}$. Multiply the guessing by t for the same argument gives

$U = Ate^{3t}$, $U' = 3Ate^{3t} + Ae^{3t}$ and $U'' = 9tAe^{3t} + 6Ae^{3t}$

Putting U and its derivatives in (37) gives

$$9tAe^{3t} + 6Ae^{3t} - 15Ate^{3t} - 5Ae^{3t} + 6Ate^{3t} = 2e^{3t} \quad (38)$$

Leading to $A = 2$. Therefore $U = 2te^{3t}$, $Y(t) = Ue^{\alpha t}$

$$Y(t) = 2te^{3t} \quad (39)$$

Method of De Oliveira for Example 2

$$y'' - 5y' + 6y = 2e^{3t} \quad (40)$$

Let $Y(t) = qe^{\alpha t}$. Considering the left hand side of (40), $\alpha = 0$

Using $\frac{p''q''}{2!} + \frac{p'q'}{1!} + pq = Q_t$ (41)

Where Q_t is real, $p(\alpha) = \alpha^2 - 5\alpha + 6 = 6$, $p'(\alpha) = 2\alpha - 5 = -5$ and $p''(\alpha) = 2$

Substituting $p(\alpha)$ and its derivatives in (41) gives

$$q'' - 5q' + q = 2e^{3t} \quad (42)$$

This is similar to (Gupta, 1998), therefore multiply the guessing by t gives $q = Ate^{3t}$, $q' = 3Ate^{3t} + Ae^{3t}$, $q'' = 9Ate^{3t} + 6Ae^{3t}$ putting these into (42) yields

$$9Ate^{3t} + 6Ae^{3t} - 15Ate^{3t} - 5Ae^{3t} + Ate^{3t} = 2e^{3t} \quad (43)$$

Which leads to $A = 2$. Therefore, the solution becomes

$$Y(t) = 2te^{3t} \quad (44)$$

Method of Jia and Sogabe for Example 2

$$y'' - 5y' + 6y = 2e^{3t} \quad (45)$$

This also shows that $\alpha = 3$ and $p(\alpha) = \alpha^2 - 5\alpha + 6$, $p(\alpha) = 0$ implying α is the root of the characteristics equation of the homogeneous part of the given differential equation.

The derivative becomes

$p'(\alpha) = 2\alpha - 5$, $p'(\alpha) = 1$ so $k = 1$ and $p''(\alpha) = 2$

Then $(t) = \frac{f(t)k!}{p^k(\alpha)}$, $g(t) = \frac{f(t)}{p'(\alpha)}$, $g(t) = 2$ and $g'(t) = 0$.

Our $U^k(t) = \sum_{j=0}^{m} d_i g^i(t)$.

And its derivative

$U'(t) = g(t)d_0$. Also $d_o = 1$ and $d_i = -\frac{k!}{p'(\alpha)}\sum_{j=0}^{m}\frac{p^{(i+k+j)}(\alpha)}{(i+k+j)}d_0$ then $d_1 = \frac{1}{2}$

Therefore $U'(t) = 2$, integrating gives $U = 2t$ which leads to the final solution as

$$Y(t) = 2te^{3t} \quad (46)$$

Example 3: Consider

$$y'' - 2y' + y = x^2 \quad (47)$$

The homogenous solution is given as

$$y_h = Ae^x + Be^{-x} \quad (48)$$

Method of Undetermined Coefficient for Example 3

Let $Y(t) = Cx^2 + Dx + E$, then $Y'(t) = 2Cx + D$ and $Y''(t) = 2C$

Inserting $Y(t)$ and its derivatives in (47) gives

$$2c - 2(2Cx + D) + (Cx^2 + Dx + E) = x^2 \quad (49)$$

Solving the above equation gives $C = 1$, $D = 4$ and $E = 6$ which gives results the solution as

$$Y(t) = x^2 + 4x + 6 \quad (50)$$

Method of Gupta for Example 3

$$y'' - 2y' + y = x^2 \quad (51)$$

Let $Y(t) = ue^{\alpha x}$, $Y'(t) = u'e^{\alpha x} + u \propto e^{\alpha x}$ and $Y''(t) = u''e^{\alpha x} + 2u' \propto e^{\alpha x} u \propto^2 e^{\alpha x}$.

Substituting $Y(t)$ and its derivatives in (51) gives

$$u''e^{\alpha x} + 2u' \propto e^{\alpha x} u \propto^2 e^{\alpha x} - 2(u'e^{\alpha x} + u \propto e^{\alpha x}) + ue^{\alpha x} = x^2 \quad (52)$$

Then (53) becomes

$$u'' - 2u' + u = x^2 \quad (53)$$

Let $u = Ax^2 + Bx + c$, then $u' = 2Ax + B$ and $u'' = 2A$ putting u and its derivatives in (53) gives

$$2A - 4Ax - 2B + Ax^2 + Bx + C = x^2 \quad (54)$$

Solving the above equation gives, $A = 1$, $B = 4$ and $C = 6$, $u = x^2 + 4x + 6$ and $Y(t) = ue^{\alpha x}$

$$Y(t) = x^2 + 4x + 6 \quad (55)$$

Method of De Oliveira for Example 3

$$y'' - 2y' + y = x^2 \quad (56)$$

The characteristics equation is $p(\lambda) = \lambda^2 - 2\lambda + 1$. Then $p'(\lambda) = 2\lambda - 2$ and $p''(\lambda) = 2$

$\frac{p''q''}{2!} + \frac{p'q'}{1!} + pq = x^2$, $\frac{2q''}{2!} - \frac{2q'}{1!} + q = x^2$ and

$$q'' - 2q' + q = x^2 \quad (57)$$

Let $q = Ax^2 + Bx + C$then $q' = 2Ax + B$ and $q'' = 2A$ putting these in (57) gives

$$2A - 2(2Ax + B) + (Ax^2 + Bx +) = x^2 \quad (58)$$

Which lead to $A = 1$, $B = 4$, $C = 6$ and $y = ue^{\alpha x}$

$$Y(t) = x^2 + 4x + 6 \quad (59)$$

Method of Jia and Sogabe for Example 3

$$y'' - 2y' + y = x^2 \quad (60)$$

Here $f(x) = x^2$, $\lambda = 0$ we have $d_o = 1$ and $d_i = \frac{-1}{p(\lambda)} \sum_{j=0}^{i-1} \frac{p^{(i+-j)}(\lambda)}{(i+-j)!} d_j$

The characteristics equation is $p(\lambda) = \lambda^2 - 2\lambda + 1$. Then $p'(\lambda) = 2\lambda - 2$ and $p''(\lambda) = 2$

Let $Y(t) = u(x)e^{\alpha x}$

$$d_1 = \frac{-1}{p(\lambda)}\left(\frac{p'(\lambda)}{1!}\right)d_o \text{ therefore } d_1 = \left(\frac{-1}{1}\right)\left(\frac{-2}{1!}\right)(1) = 2 \quad (61)$$

$$d_2 = \frac{-1}{p(\lambda)}\left(\left(\frac{p''(\lambda)}{2!}\right)d_o + \left(\frac{p'(\lambda)}{1!}\right)d_1\right) \text{ therefore } d_2 = \left(\frac{-1}{1}\right)\left(\left(\frac{2}{2!}\right)(1) + \left(\frac{-2}{2}\right)(2)\right) = 3$$

(62)

$$g(x) = \frac{f(x)}{p(\lambda)} \text{ by substituting back } g(x) = \frac{x^2}{1} = x^2 \quad (63)$$

$$u(x) = \sum_{i=0}^{m} d_i g^i(x) \text{ then } u(x) = d_o g(x) + d_1 g'(x) + d_2 g''(x)$$

(64)

$$u(x) = x^2 + 4x + 6 \quad (65)$$

Therefore

$$Y(t) = x^2 + 4x + 6 \quad (66)$$

Example 4: Consider

$$y'' + 4y = \cos 2t \quad (67)$$

The homogenous solution is

$$y_h = A\cos 2t + B\sin 2t \quad (68)$$

Method of Undetermined Coefficient for Example 4

$$Y(t) = tC\cos 2t + Dt\sin 2t \quad (69)$$

$$Y'(t) = C\cos 2t - 2Ct\sin 2t + D\sin 2t + 2D\cos 2t \quad (70)$$

$$Y''(t) = -4c\sin 2t - 4ct\cos 2t + 4D\cos 2t - 4D\sin 2t \quad (71)$$

Inserting (71) and (69) in (67) gives

$$-4c\sin 2t - 4ct\cos 2t + 4D\cos 2t - 4D\sin 2t - 4C\sin 2t + 4D\cos 2t = \cos 2t \quad (72)$$

Solving the above equation we have $D = \frac{1}{4}$, $C = 0$ therefore

$$Y(t) = \frac{t}{4}\sin 2t \quad (73)$$

Method of Gupta for Example 4

$$y'' + 4y = \cos 2t = e^{2it} \quad (74)$$

Let $Y(t) = ue^{\alpha x}$, $Y'(t) = u'e^{\alpha x} + u \propto e^{\alpha x}$ and $y'' = u''e^{\alpha x} + 2u' \propto e^{\alpha x} u \propto^2 e^{\alpha x}$

Substituting $Y(t)$ and its derivatives in (74) gives

$$u'' + 4u' = 1 \quad (75)$$

Let $v = u'$, $v' = u''$, $v = At + B$ and $v' = A$.

Putting these in (75) gives $A + 4i(At + B) = 1$ therefore $B = -\frac{i}{4}$ and $A = 0$.

Then $v = -\frac{it}{4}$, $Y(t) = ue^{\propto t}$, $Y(t) = -\frac{it}{4}(\cos 2t + i\sin 2t)$

$$Y(t) = \frac{t}{4}\sin 2t \quad (76)$$

Method of De Oliveira for Example 4

$$y'' + 4y = \cos 2t = e^{2it} \quad (77)$$

The characteristics equation is $p(\lambda) = \lambda^2 + 4$. $p'(\lambda) = 2\lambda$ and $p''(\lambda) = 2$

Therefore $\frac{p''q''}{2!} + \frac{p'q'}{1!} + pq = 1$, $\frac{2q''}{2!} + \frac{4iq'}{1!} = 1$ and

$$q'' + 4iq' = 1 \quad (78)$$

The Same case with (Gupta) method

Let $v = q'$ and $v' = q''$. Therefore we have $v' + 4iv = 1$, $v = At + B$ and $v' = A$

Substituting these in (78) gives $A + 4i(At + B) = 1$. Which yield $B = \frac{-i}{4}$ and $A = 0$

Then $v = \frac{-it}{4}$ and $Y(t) = ue^{\alpha x} = -\frac{i}{4}(\cos 2t + \sin 2t)$

$$Y(t) = \frac{t}{4}\sin 2t \quad (79)$$

Method of Jia and Sogabe for Example 4

$$y'' + 4y = \cos 2t = e^{2it} \quad (80)$$

In this case $\lambda = 2i$, $p(\lambda) = \lambda^2 + 4$, $p'(\lambda) = 2\lambda$, $p''(\lambda) = 2$ and $k = 1$. Then $d_o = 1$

$u^{(k)}(t) = \sum_{i=0}^{m} d_i g^i(t)$ and $u'(t) = d_o g(t) = -\frac{i}{4}$. Then $Y(t) = ue^{2it} = \frac{-it}{4}(\cos 2t + i\sin 2t)$

$$Y(t) = \frac{t}{4}\sin 2t \quad (81)$$

Example 5: Consider

$$y''' - 5y'' + 3y' + 9y = te^{3t} \quad (82)$$

The homogeneous solution is given as

$$y_h = c_1e^{3t} + c_2te^{3t} + c_3e^{-t} \quad (83)$$

Method of Undetermined Coefficient for Example 5

For the particular solution using the method of undetermined coefficient we guess

$$Y(t) = (At^3 + Bt^2)e^{3t} \quad (84)$$

Differentiation of $Y(t)$ gives

$$Y'(t) = (3At^2 + 2Bt)e^{3t} + 3(At^3 + Bt^2)e^{3t} \quad (85)$$

$$Y''(t) = (6At + 2B)e^{3t} + 6(3At^2 + 2Bt)e^{3t} + 9(At^3 + Bt^2) \quad (86)$$

And $Y'''(t) = 6Ae^{3t} + 9(6At + 2B)e^{3t} + 27(3At^2 + 2Bt)e^{3t} + 27(At^3 + Bt^2)$ (87)

Substituting (87), (86) and (85) in (82) gives

$$6Ae^{3t} + 4(6At + 2B)e^{3t} = te^{3t} \quad (88)$$

Solving (88) gives $A = \frac{1}{24}$ and $B = -\frac{1}{32}$

The particular solution is

$$Y(t) = (\frac{1}{24}t^3 - \frac{1}{32}t^2)e^{3t} \quad (89)$$

Method of Gupta for Example 5

$$y''' - 5y'' + 3y' + 9y = te^{3t} \quad (90)$$

We let $Y(t) = ue^{\alpha t}$ where u is a polynomial. Differentiation of $Y(t)$ gives

$$Y'(t) = u'e^{\alpha t} + \alpha ue^{\alpha t} \quad (91)$$

$$Y''(t) = u''e^{\alpha t} + 2\alpha u'e^{\alpha t} + \alpha^2 ue^{\alpha t} \quad (92)$$

And

$$Y'''(t) = u'''e^{\alpha t} + 3\alpha u''e^{\alpha t} + 3\alpha^2 u'e^{\alpha t} + \alpha^3 ue^{\alpha t} \quad (93)$$

Substituting $Y(t)$ and its derivatives in (90) gives

$$(u''' + 3\alpha u'' + 3\alpha^2 u' + \alpha^3 u)e^{3t} - 5(u'' + 2\alpha u' + \alpha^2 u)e^{3t} + 3(u' + \alpha u)e^{3t} + 9ue^{3t} = te^{3t} \quad (94)$$

With $\alpha = 3$, (94) becomes

$$u''' + 4u'' = t \quad (95)$$

Let $u = At^3 + Bt^2$, derivatives of u gives $u' = 3At^2 + 2Bt$, $u'' = 6At + 2B$, and $u''' = 6A$

Substituting u''' and u'' in (95) gives $6A + 4(6At + 2B) = t$ yielding $A = \frac{1}{24}$ and $B = -\frac{1}{32}$

Then the particular solution is

$$Y(t) = ue^{\alpha t} = (\frac{1}{24}t^3 - \frac{1}{32}t^2)e^{3t} \quad (96)$$

Method of De Oliveira for Example 5

$$y''' - 5y'' + 3y' + 9y = te^{3t} \quad (97)$$

The characteristic polynomial is

$p(\lambda) = \lambda^3 - 5\lambda^2 + 3\lambda + 9$, $p'(\lambda) = 3\lambda^2 - 10\lambda + 3$, $p''(\lambda) = 6\lambda - 10$ and $p'''(\lambda) = 6$

Inserting $p(\lambda)$ and its derivatives in the following equation

$$\left[\frac{p^{(n)}(\lambda)}{n!}q^{(n)} + \cdots + \frac{p''(\lambda)}{2!}q'' + \frac{p'(\lambda)}{1!}q' + p(\lambda)q\right]e^{\lambda t} = f(t)$$

Gives

$$[q''' + (3\lambda - 5)q'' + (3\lambda^2 - 10\lambda + 3)q' + (\lambda^3 - 5\lambda^2 + 3\lambda + 9)q]e^{\lambda t} = te^{3t} \quad (98)$$

Clearly we can see that $\lambda = 3$, the above equation becomes

$$q''' + 4q'' = t \quad (99)$$

Let $q = At^3 + Bt^2$, differentiation of q gives $q' = 3At^2 + 2Bt$, $q'' = 6At + 2B$ and $q''' = 6A$

Substituting q''' and q'' in (99) gives $6A + 4(6At + 2B) = t$ Solving this equation gives $A = \frac{1}{24}$ and $B = -\frac{1}{32}$

Then the particular solution is

$$Y(t) = qe^{\lambda t} = (\frac{1}{24}t^3 - \frac{1}{32}t^2)e^{3t} \quad (100)$$

Method of Jia and Sogabe for Example 5

$$y''' - 5y'' + 3y' + 9y = te^{3t} \quad (101)$$

Set $p(\lambda) = \lambda^3 - 5\lambda^2 + 3\lambda + 9$, $f(t) = t$, $\alpha = 3$. Since $p(\alpha) = p'(\alpha) = 0$ and $p'' \neq 0$, we have $k = 2$

Set $g(t) = f(t) \cdot \frac{k!}{p^{(k)}(\alpha)} = \frac{1}{4}t \quad (102)$

Set $d_o = 1$ and $d_i = -\frac{k!}{p^{(k)}(\alpha)}\sum_{j=0}^{i-1}\frac{p^{(i+k-j)}(\alpha)}{(i+k-j)!}d_j$

By using d_i, we get $d_1 = -\frac{2!}{p''(3)} \cdot \frac{p'''(3)}{3!}d_o = -\frac{1}{4} \quad (103)$

Then set $u^{(k)}(t) = \sum_{i=0}^{m} d_i g^{(i)}(t)$, $u''(t) = \sum_{i=0}^{1} d_i g^{(i)}(t) = \frac{1}{4}t - \frac{1}{16} \quad (104)$

Hence $u(t) = \frac{1}{24}t^3 - \frac{1}{32}t^2$. Finally, we obtain the particular solution from

$$Y(t) = u(t)e^{\alpha t} = (\frac{1}{24}t^3 - \frac{1}{32}t^2)e^{3t} \quad (105)$$

Example 6: Consider

$$y'' - 2y' + 2y = te^t \sin 3t \quad (106)$$

The homogenous solution is given as

$$y_h = Ae^t \sin t + Be^t \sin t \quad (107)$$

Method of Undetermined Coefficient for Example 6

Guessing $Y(t) = (At + B)e^{t(1+3i)}$ (108)

$Y'(t) = Ae^{t(1+3i)} + (At + B)(1 + 3i)e^{t(1+3i)}$ (109)

and $Y''(t) = 2(1 + 3i)Ae^{t(1+3i)} + (At + B)(1 + 3i)(1 + 3i)e^{t(1+3i)}$ (110)

Inserting (110), (109) and (108) in (106) gives

$2(1 + 3i)Ae^{t(1+3i)} + (At + B)(1 + 3i)(1 + 3i)e^{t(1+3i)} - 2Ae^{t(1+3i)} - 2(At + B)(1 + 3i)e^{t(1+3i)} + 2(At + B)e^{t(1+3i)} = te^{t(1+3i)}$

(111)

Solving (111) gives $6iA - 8(At + B) = t$, $A = -\frac{1}{8}$ $and\ B = -\frac{3i}{32}$

$Y(t) = \left(-\frac{t}{8} - \frac{3i}{32}\right)e^{t(1+3i)} = \left(-\frac{t}{8} - \frac{3i}{32}\right)e^{t}(\cos 3t + i\sin 3t)$ (112)

The particular solution is

$Y(t) = -\frac{t}{8}e^{t}\cos 3t + \frac{3}{32}e^{t}\sin 3t$ (113)

Method of Gupta for Example 6

$y'' - 2y' + 2y = te^{t}\sin 3t = te^{t(1+3i)}$ (114)

We have $\alpha = (1 + 3i)$. Let $Y(t) = Ue^{\alpha t}$, $Y'(t) = U'e^{\alpha t} + U\alpha e^{\alpha t}$ and

$Y''(t) = U''e^{\alpha t} + 2U'\alpha e^{\alpha t} + U\alpha^2 e^{\alpha}$

Substituting $Y(t)$ and its derivatives in (114) gives

$U''e^{\alpha t} + 2U'\alpha e^{\alpha t} + U\alpha^2 e^{\alpha} - 2U'e^{\alpha t} - 2U\alpha e^{\alpha t} + 2Ue^{\alpha t} = te^{t(1+3i)}$ (115)

$U'' + 6iU' - 8U = t$

(116)

Guessing $U = At + B$, $U' = A$ and $U'' = 0$. (116) becomes $0 + 6it - 8At - 8B = t$ which yield $A = -\frac{1}{8}$ $and\ B = -\frac{3i}{32}$. Therefore $U = -\frac{t}{8} - \frac{3i}{32}$ and

$Y(t) = \left(-\frac{t}{8} - \frac{3i}{32}\right)e^{t}(\cos 3t + i\sin 3t)$ (117)

The particular solution is

$Y(t) = -\frac{te^{t}\cos 3t}{8} + \frac{3e^{t}\sin 3t}{32}$ (118)

Method of De Oliveira for Example 6

$y'' - 2y' + 2y = te^{t}\sin 3t$ (119)

Let $Y(t) = qe^{\alpha t}$ then $Y(t) = qe^{t(1+3i)}$ And $since\ \alpha = (1 + 3i)$

The characteristic equation is $p(\alpha) = \alpha^2 - 2\alpha + 2 = -8$, $p'(\alpha) = 2\alpha - 2 = 6i$ and $p''(\alpha) = 2$

Let $\frac{p''q''}{2!} + \frac{p'q'}{1!} + pq = Q(t)$

$q'' + 6iq' - 8q = t$ (120)

Guessing $q = At + B$, $q' = A$ and $q'' = 0$.putting these in (120) gives $0 + 6iA - 8At - 8B = t$ which yield $A = -\frac{1}{8}$ $and\ B = -\frac{3i}{32}$

Hence $q = -\frac{t}{8} - \frac{3i}{32}$ and $Y(t) = \left(-\frac{t}{8} - \frac{3i}{32}\right)e^{t(1+3i)}$

$Y(t) = \left(-\frac{t}{8} - \frac{3i}{32}\right)(e^{t}\cos 3t + i\sin 3t)$ (121)

The particular solution is

$Y(t) = -\frac{t}{8}e^{t}\cos 3t + \frac{3}{32}e^{t}\sin 3t$ (122)

Method of Jia and Sogabe for Example 6

$y'' - 2y' + 2y = te^{t}\sin 3t = te^{t(1+3i)}$ (123)

The characteristic equation is $p(\alpha) = \alpha^2 - 2\alpha + 2$, $but\ \alpha = (1 + 3i) there\ fore$, $p(\alpha) = -8$

$p'(\alpha) = 2\alpha - 2 = 2(1 + 3i) - 2$, $p'(\alpha) = 6i$ and $p''(\alpha) = 2$

$g(t) = \frac{f(t)k!}{p^k(\alpha)}$. Since α is not root of the equation therefore k will disappear and
$f(t) = real = t$
$g(t) = \frac{f(t)}{p(\alpha)} = -\frac{t}{8}$, $g'(t) = -\frac{1}{8}$ and $U^K(t) = \sum_{i=0}^{m} d_i g^i(t)$
Where $i = 0,1,2,3 \ldots \ldots \ldots m$, $U(t) = d_0 g(t) + d_1 g'(t)$ and $d_0 = 1$

$$d_1 = -\frac{k!}{p^k(\alpha)} \sum_{j=0}^{m} \frac{p^{(i+j+k)}}{(i+j+k)}$$

$$d_1 = -\frac{1}{p(\alpha)} p'(\alpha) = \frac{3i}{4} \quad (124)$$

$$U(t) = -\frac{t}{8} - \frac{3i}{32} \quad (125)$$

$$Y(t) = \left(-\frac{t}{8} - \frac{3i}{32}\right) e^t(\cos 3t + i \sin 3t) \quad (126)$$

The particular solution is

$$Y(t) = -\frac{t}{8} e^t \cos 3t + \frac{3}{32} e^t \sin 3t \quad (127)$$

Results and Discussion

Table 1: Rating of the method performances

Examples	Method Of Undetermined Coefficient	Gupta Method (Hinton, 1994)	De Oleviera Method (Oliveira, 2012)	Jia And Sogabe Method (Jia & Sogabe, 2013)
1) $y'' + y = tsint$	Lengthy	moderate	moderate	faster
2) $y'' - 5y' + 6y = 2e^{3t}$	Faster	moderate	moderate	moderate
4) $y'' + 4y = \cos 2t$	Lengthy	moderate	moderate	faster
5) $y''' - 5y'' + 3y' + 9y = te^{3t}$	Lengthy	moderate	Moderate	faster
6) $y'' - 2y' + 2y = te^t \sin 3t$	Lengthy	moderate	moderate	faster

Conclusion

In this paper we investigated the strengths and weaknesses of four methods of obtaining the particular solution of non-homogeneous ordinary differential equation. We have realized that the method of undetermined coefficient is the best often when the inhomogeneous function is only exponential function. (Jia & Sogabe, 2013) method is found to be simpler in most cases and the best when the degree of differential equation is more than two.

References

Boyce, W. E., & DiPrima, R. C. (1992). Elementary differential equations and boundary value problems. *Newyork Wiley, 184-190.*

De Leon, D. (2010). Euler-Cauchy Using Undetermined Coefficients. *The College Mathematics Journal*. http://doi.org/10.4169/074683410X488728

Gupta, R. (1998). Classroom Note: On Particular Solutions of Linear Difference Equations with Constant Coefficients. *SIAM Rev.,vol. 40, no. 3, 680-684.*

Jia, J., & Sogabe, T. (2013). On particular solution of ordinary differential equations with constant coefficients. *Applied Mathematics and Computation*, *219*(12), 6761–6767. http://doi.org/10.1016/j.amc.2012.12.080

Leon, D. De. (2015). Using Undetermined Coefficients to Solve Certain Classes of Variable-Coefficient.

Oliveira, O. R. B. De. (2012). A formula substituting the undetermined coefficients and the annihilator methods. *International Journal of Mathematical Education in Science and Technology*. http://doi.org/10.1080/0020739X.2012.714496

Ortigueira, M. D. (2014). A simple approach to the particular solution of constant coefficient ordinary differential equations. *Applied Mathematics and Computation*, *232*, 254–260. http://doi.org/10.1016/j.amc.2014.01.036