STABILITY ANALYSIS OF ENDEMIC EQUILIBRIUM OF A LASSA FEVER MODEL

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Abstract
A Mathematical Model was developed for the spread and control of Lassa Lever. The disease free and endemic equilibrium states were obtained and analysed for stability. Key to the analysis is the basic reproductive number (R_0), *which is an important threshold for disease control. The analysis showed that the endemic equilibrium points* E_1 *is locally asymptotically stable for* R_0 *close to 1, and the bifurcation at* $p^* = 0$ *is subcritical when* $a > 0$

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Introduction
Lassa fever is an acute viral Hemorrhagic fever (VHF) first isolated in a town called Lassa in the Yedseram River Valley in the present Borno State of Northern Nigeria in 1969 (Tara, 2004). Lassa fever is endemic in Nigeria, Liberia, Sierra Leone, Guinea, and other West African countries, affecting about 2 – 3 million persons with 5000 - 10,000 fatalities annually (McCormick et al., 1987). Since its initial discovery in Lassa-Nigeria, rural and nosocomial outbreaks of Lassa fever have occurred repeatedly in other parts of Nigeria: Jos, Onitsha, Zonkwa, Ekpoma (Tomori et al., 1988).

In 2012, 397 cases with 40 deaths were recorded cutting across 12 states of Nigeria (Healthmap, 2012). More recently, between August 2015 and 23 January 2016, 159 suspected cases of Lassa fever including 82 deaths, were reported across 19 states in Nigeria (WHO, 2016).

Promed (2006) reported outbreaks in some cities of West African countries of Sierra Leone, Liberia, Guinea. Lassa fever therefore appears to have 2 geographically separate endemic areas: The Mano River region (Guinea, Sierra Leone, and Liberia) in the West, and Nigeria in the East.

Lassa fever is a zoonotic disease, i.e., it can be transmitted from infected animal to a human. The natural Reservoir of the Lassa virus is Multimammate Rat species known as Mastomys Natalencesis (Fisher-Hoch et al., 1995). Because certain varieties of *Mastomys* often live in human homes, the virus is easily transmitted to humans. Transmission occurs via direct contact with rat urine, faeces, and saliva; via contact with excretion- or secretion-infected materials; or via ingestion of excretion-contaminated food. Victims can also become infected via skin breaks, and via mucous membranes from aerosol transmission from dust-borne particles. In some areas, the rodents are used as a food source, thus providing additional exposure to the infected rat blood, as well as allowing ingestion of potentially contaminated meat. Eze et al., (2010) stated that Health workers become infected usually from contact with rodent saliva or contamination of needles.

Unlike other arena viruses, Lassa virus can be fairly easily transmitted from human to human (WHO, 2004). Richmond (2003) stated that humans can contact the disease from other humans via aerosol transmission (coughing), or from direct contact with infected

human blood, urine, or semen. Lassa virus has been isolated from semen 6 weeks after acute illness; thus the virus can be transmitted to sexual partners by convalescent men (Tara, 2004).

The symptoms of Lassa fever develop about 21 days after infection with acute illness involving multi organs. Specific symptoms include fever, facial swelling, muscle fatigue, vomiting, cough, meningitis, and hypertension. In some patients' neurological problems, including hearing loss which may be transient or permanent, tremors, and encephalitis, have been described the (Omilabu et al., 2005).

Literature

Okuonghae and Okuonghae (2006) formulated an SIS model coupled to a population of rat species, for the transmission of Lassa fever disease. They obtained the equilibrium states of their model and examined them for endemic and epidemic situations. Further, they calculated the basic reproductive number for their model and gave conditions for disease outbreak. Ogabi, et al., (2012) developed an SIR model for controlling Lassa fever transmission in northern part of Edo state, Nigeria. They advocated for health policies that will keep the basic reproductive number R_0 below 1, thereby keeping the transmission of the disease under control.

The Lassa fever model developed by (Bawa et al., 2013) is a major shift from the first two papers cited. The researchers divided the human population into susceptible human S_H, the Infected human I_H, the reservoir population they divided into Infant I_R and the Adult reservoir A_R and interestingly represented the virus in the environment by V. They explained that the virus compartment is generated from the urine and faeces of infected Human and adult reservoirs. The major parameters of their model are b_H per capital birth rate of Human, b_R per capital birth rate of the reservoir, μ_R per capital natural death rate of Human, μ_H per capital death rate of the reservoir, δ_H Lassa fever induced death rate, δ_R mortality death of the reservoir due to hunting, β_1 effective contact rate for human, β_2 effective contact rate between reservoir and human, γ recovery rate of Infected human and σ progression rate from Infant to adult reservoir. They recommended that efforts should be made to keep the basic reproductive number below unity to ensure that the virus is contained. Tolulope et al., (2015) complemented the work of Bawa et al., (2013) by introducing the quarantine parameter (I_Q) and assume that the virus confers permanent immunity to the sufferers upon recovery. The rest of their parameters are the same with that of Bawa et al., (2013). James (2015) developed a mathematical model of Lassa fever using three ordinary differential equations; they discovered that the zero equilibrium state is stable when the birth rate of the human population is less than the death rate. Their analysis also gave the condition for the non-zero equilibrium to be unstable.

Onuorah et al. (2016a) developed a Lassa fever model using the sex structure approach. Their model represented the transmission dynamics of the Lassa fever disease using a set of ordinary differential equations. The total human population at time t denoted by $N_H(t)$ was sub-divided into four (4) mutually exclusive sub-populations of Susceptible Male $S_1(t)$, Infected Male $I_1(t)$, Susceptible Female $S_2(t)$, Infected Female $I_2(t)$, such that $N_H(t) = S_1(t) + I_1(t) + S_2(t) + I_2(t)$. Similarly, the total Natural Reservoir/host population at time t, denoted by $N_R(t)$ was sub-divided into dormant Reservoir $R_1(t)$, active Reservoir $R_2(t)$, such that $N_R(t) = R_1(t) + R_2(t)$. Their model had the following

assumptions. Susceptible individuals, male/female can be infected via interaction with the active Reservoir (Mastomys Natelensis), and via sexual interaction with opposite sex. Two major controls were considered, the use of condom to reduce contact via sexual interaction and the use of pesticide or Rat poison to kill the natural Reservoir (Mastomys Natelensis). And finally, horizontal transmission for human and vertical transmission for the Reservoir.

Onuorah et al. (2016b), is an extension of Onuorah et al. (2016a), specifically, they included a schematic diagram, sensitivity analysis, numerical computation of the basic Reproductive number R_0 and numerical simulation. All the works cited above did not consider the endemic equilibrium (equilibrium state where at least one of the infected compartments is non-zero) of their Lassa fever model. In this work we intend to bridge this gap identified, by extending the analysis of Onuorah et al (2016a) to endemic equilibrium states of the various state variables of our model. We also carried out bifurcation analysis.

Methodology

Parameters of the Model

β_H The natural birth rate of human population

β_R The natural birth rate of vectors

θ The proportion of human birth that is male $0 < \theta < 1$

ρ Spectral Radius

α_1 The rate of transmission resulting from sexual interaction between infected female and susceptible male

α_2 The rate of transmission resulting from sexual interaction between infected male and susceptible female

α_3 The rate of transmission resulting from interaction between active virus Reservoir and susceptible male

α_4 The rate of transmission resulting from interaction between active virus Reservoir and susceptible female

c_1 Average number of male partners acquired by a susceptible female

c_2 Average number of female partners acquired by a susceptible male

μ_1 Natural death rate of human population

μ_2 Natural death rate of Reservoir population

γ Recovery rate of infected human

σ Progression rate from dormant to active Reservoir host

δ_1 Death rate of human population due to infection

δ_2 Death rate of Reservoir population due to application of pesticide

ε Efficacy of condom

τ Compliance of condom usage

The Model

From the assumptions above we have the following equations:

$$\frac{dS_1}{dt} = \beta_H \theta N_H + \gamma I_1 - \frac{(c_2\alpha_1(1-\varepsilon\tau)I_2 + \alpha_3 R_2)S_1}{N_H} - \mu_1 S_1 \quad (1)$$

$$\frac{dI_1}{dt} = \frac{(c_2\alpha_1(1-\varepsilon\tau)I_2 + \alpha_3 R_2)S_1}{N_H} - (\mu_1 + \delta_1 + \gamma)I_1 \quad (2)$$

$$\frac{dS_2}{dt} = \beta_H(1-\theta)N_H + \gamma I_2 - \frac{(c_1\alpha_2(1-\varepsilon\tau)I_1 + \alpha_4 R_2)S_2}{N_H} - \mu_1 S_2 \quad (3)$$

$$\frac{dI_2}{dt} = \frac{(c_1\alpha_2(1-\varepsilon\tau)I_1 + \alpha_4 R_2)S_2}{N_H} - (\mu_1 + \delta_1 + \gamma)I_2 \quad (4)$$

$$\frac{dR_1}{dt} = \beta_R N_R - (\sigma + \mu_2 + \delta_2)R_1 \quad (5)$$

$$\frac{dR_2}{dt} = \sigma R_1 - (\mu_2 + \delta_2)R_2 \quad (6)$$

The total human population size is given by;

$$N_H = S_1 + I_1 + S_2 + I_2 \quad (7)$$

The total Reservoir population size is given by

$$N_R = R_1 + R_2 \quad (8)$$

By adding equations (1) to (4), we have;

$$\frac{dN_H}{dt} = \beta_H N_H - \mu_1 N_H - \delta_1(I_1 + I_2) \quad (9)$$

By adding equations (5) to (6), we have;

$$\frac{dN_R}{dt} = (\beta_R - \mu_2 + \delta_2)N_R \quad (10)$$

Basic Properties of the Model

In this section, the basic dynamical features of the model equations (1) to (6) will be explored.

Theorem 1 The closed set

$$D = \left\{(S_1, I_1, S_2, I_2, R_1, R_2) \in \Re_+^6 : S_1 + I_1 + S_2 + I_2 \le N_H; R_1 + R_2 \le N_H\right\}$$

Is positively-invariant and attracting with respect to the basic model equations (1) to (6)

Proof

From equations (7), to (10);

$\frac{dN_H}{dt} \le (\beta_H - \mu_1)N_H$, $\frac{dN_R}{dt} \le (\beta_R - \mu_2 - \delta)N_R$.

It follows that $\frac{dN_H}{dt} < 0$ and $\frac{dN_R}{dt} < 0$ if $N_H(t) > \frac{\beta_H}{\mu_1}$ and $N_R(t) > \frac{\beta_R}{\mu_2}$ respectively. Thus a standard comparison theorem as in (Lakshmickantham et al, 1999) can be used to show that $N_H(t) \le N_H(0)e^{\mu_1(t)} + \frac{\beta_H}{\mu_1}\left(1 - e^{-\mu_1(t)}\right)$

and $N_R(t) \le N_R(0)e^{\mu_2(t)} + \frac{\beta_r}{\mu_2 + \delta_2}\left(1 - e^{(-\mu_2+\delta_2)(t)}\right)$. In particular $N_H(t) \le \frac{\beta_H}{\mu_1}$ and $N_R(t) \le \frac{\beta_R}{\mu_2 + \delta_2}$ if $N_H(0) \le \frac{\beta_H}{\mu_1}$ and $N_R(0) \le \frac{\beta_R}{\mu_2 + \delta_2}$ respectively. Thus D is positively-invariant. Further, if $N_H(0) > \frac{\beta_H}{\mu_1}$, and $N_R(0) > \frac{\beta_R}{\mu_2 + \delta_2}$, then either the solution enters

D in finite time or $N_H(t)$ approaches $\frac{\beta_H}{\mu_1}$, and $N_R(t)$ approaches $\frac{\beta_R}{\mu_2+\delta_2}$, and the infected variables I_1+I_2, R_1+R_2 approaches 0. Hence D is attracting, that is all solutions in $\Re_+^6$ eventually enters D. Thus in D, the basic model equations (1) to (6) is well posed epidemiologically and mathematically according to (Hethcote, 1978). Hence it is sufficient to study the dynamics of the basic model equations (1) to (6)

Disease Free Equilibrium (DFE)

At equilibrium states, the rate of change of the state variables with respect to time is zero, i.e.

$$\frac{dS_1}{dt}=\frac{dI_1}{dt}=\frac{dS_2}{dt}=\frac{dI_2}{dt}=\frac{dR_1}{dt}=\frac{dR_2}{dt}=0$$

We define disease compartments as the Infected male, Infected female compartments that is I_1 and I_2. we let $(S_1,I_1,S_2,I_2,R_1,R_2)=(x,y,z,u,v,w)$ at disease free equilibrium, equating the right hand side of our model equation (1) to (6) to zero and solving with the above change of variable, we have our DFE

$$E_0=(x,y,z,u,v,w)=\left(\frac{\beta_H\theta N_H}{\mu_1},0,\frac{\beta_H(1-\theta)N_H}{\mu_1},0,0,0\right) \quad (11)$$

For the analysis of Local Stability and is Globally Asymptotically stable of Disease Free Equilibrium E_0 of the model, the reader is referred to Onuorah (2016b)

At DFE, the Jacobian matrix is

$$J_{E_0}=\begin{bmatrix} -\mu_1 & \gamma & 0 & \frac{px}{N_H} & 0 & \frac{\alpha_3 x}{N_H} \\ 0 & -A_1 & 0 & \frac{px}{N_H} & 0 & \frac{\alpha_3 x}{N_H} \\ 0 & \frac{qz}{N_H} & -\mu_1 & 0 & 0 & \frac{\alpha_4 z}{N_H} \\ 0 & \frac{qz}{N_H} & 0 & -A_1 & 0 & \frac{\alpha_4 z}{N_H} \\ 0 & 0 & 0 & 0 & -A_2 & 0 \\ 0 & 0 & 0 & 0 & \sigma & -(\mu_2+\delta_2) \end{bmatrix} \quad (12)$$

where $p=c_2\alpha_1(1-\varepsilon\tau)$, $q=c_1\alpha_2(1-\varepsilon\tau)$ $A_1=(\mu_1+\delta_1+\gamma)$ and $A_2=(\sigma+\mu_2+\delta_2)$

$A_3=(\mu_2+\delta_2)$

Basic Reproductive Number (R_0)

We use the next generation matrix approach as described by (Driessche and Wathmough, 2005) to derive our Basic Reproductive Number diseases.

Here, the basic reproductive number R_0 is the spectral radius of the product matrix FV^{-I}, i.e. $R_0=\rho(FV^{-I})$

Our model has four Infected compartments namely the Infective male I_1, Infected female I_2, dormant Reservoir R_1 and active Reservoir R_2, It follows that the matrices F and V for the new infective terms and remaining transfer terms respectively are given below:

$$F=\begin{bmatrix} 0 & \frac{px}{N_H} & 0 & \alpha_3 x \\ \frac{qz}{N_H} & 0 & 0 & \alpha_4 z \\ 0 & 0 & \beta_R & \beta_R \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad V=\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & -\sigma & A_3 \end{bmatrix}$$

$$V^{-I}=\begin{bmatrix} \frac{1}{A_1} & 0 & 0 & 0 \\ 0 & \frac{1}{A_1} & 0 & 0 \\ 0 & 0 & \frac{1}{A_2} & 0 \\ 0 & 0 & \frac{\sigma}{A_2 A_3} & \frac{1}{A_3} \end{bmatrix}$$

$$FV^{I}=\begin{bmatrix} 0 & \frac{px}{A_1 N_H} & \frac{x\sigma\alpha_3}{A_2 A_3} & \frac{x\alpha_3}{A_3} \\ \frac{qz}{A_1 N_H} & 0 & \frac{x\sigma\alpha_4}{A_2 A_3} & \frac{x\alpha_4}{A_3} \\ 0 & 0 & \frac{\beta_R(A_3+\sigma)}{A_2 A_3} & \frac{\beta_R}{A_3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(FV^{-I})=\sqrt{\frac{px.qz}{(N_H A_1)^2}}$$

Substituting the values of x, z at equilibrium, the values of A_1, p and q gives

$$R_0=\sqrt{\frac{(c_2\alpha_1(1-\varepsilon\tau)\beta_H\theta)\times(c_1\alpha_2(1-\varepsilon\tau)\beta_H(1-\theta))}{\mu_1((\mu_1+\delta_1+\gamma))^2}} \qquad (14)$$

Endemic Equilibrium

This is an equilibrium state where at least one of the infected compartments is non-zero. In order to find the Endemic equilibrium for our model equations (1) to (6), the following steps are taken. We let $E_e=(x^*,y^*,z^*,u^*,v^*,w^*)$ represent any arbitrary point of the Endemic Equilibrium of our model equations further,

let; $\lambda_M^*=\frac{c_2\alpha_1(1-\varepsilon\tau)u^*+\alpha_3 w^*}{N_H}, \lambda_F^*=\frac{c_1\alpha_2(1-\varepsilon\tau)y^*+\alpha_4 w^*}{N_H}$ (15)

Be the force of infection of susceptible male and susceptible female respectively. Solving our model equations (1) to (6) at steady state gives

$$u^* = \frac{\beta_H(1-\theta)N_H\lambda_F}{(\lambda_F+\mu_1)(\mu_1+\delta_1+\gamma)-\gamma\lambda_F} v^* = \frac{\beta_R N_R}{(\sigma+\mu_2+\delta_2)}, w^* = \frac{\sigma\beta_R N_R}{(\mu_2+\delta_2)(\sigma+\mu_2+\delta_2)},$$

$$, \ x^* = \frac{(\mu_1+\delta_1+\gamma)\beta_H\theta N_H}{(\lambda_M+\mu_1)(\mu_1+\delta_1+\gamma)-\gamma\lambda_M}, z^* = \frac{(\mu_1+\delta_1+\gamma)\beta_H(1-\theta)N_H}{(\lambda_F+\mu_1)(\mu_1+\delta_1+\gamma)-\gamma\lambda_F}, \quad (16)$$

$$y^* = \frac{\beta_H\theta N_H\lambda_M}{(\lambda_M+\mu_1)(\mu_1+\delta_1+\gamma)-\gamma\lambda_M}$$

Substituting (16) into (15) we have;

$$\lambda_F^* = \frac{k_1 q\beta_H\theta H_H\lambda_M^* + (\alpha_4\sigma\beta_R N_R)\left[\lambda_M^*(\mu_1+\delta_1)+k_2\right]}{\left[\lambda_M^*(\mu_1+\delta_1)+k_2\right]k_1} \quad (17)$$

$$\lambda_F^* = \frac{k_3\lambda_M^* + \lambda_M\alpha_4\sigma\beta_R N_R(\mu_1+\delta_1)+k_2\alpha_4\sigma\beta_R N_R}{\lambda_M^*(\mu_1+\delta_1)k_1+k_2k_1} \quad (18)$$

$$\lambda_F^* = \frac{k_3\lambda_M^* + k_4\lambda_M^* + k_2\alpha_4\sigma\beta_R N_R}{k_5\lambda_M^*+k_2k_1} \quad (19)$$

and

$$\lambda_M^* = \frac{k_1 p\beta_H(1-\theta)N_H\lambda_F^* + (\alpha_3\sigma\beta_R N_R)\left[(\mu_1+\delta_1)\lambda_F^*+k_2\right]}{\left[(\mu_1+\delta_1)\lambda_F^*+k_2\right]k_1} \quad (20)$$

$$\lambda_M^* = \frac{k_1 p\lambda_F^* + \alpha_3\sigma\beta_R N_R(\mu_1+\delta_1)\lambda_F^* + k_2\alpha_3\sigma\beta_R N_R}{k_1(\mu_1+\delta_1)\lambda_F^*+k_2k_1} \quad (21)$$

$$\lambda_M = \frac{k_6\lambda_F^* + k_7\lambda_F^* + k_2\alpha_3\sigma\beta_R N_R}{k_8\lambda_F^*+k_2k_1} \quad (22)$$

Where $P = c_2\alpha_1(1-\varepsilon\tau), q = c_1\alpha_2(1-\varepsilon\tau), k_1 = (\mu_2+\delta_2)(\sigma+\mu_2+\delta_2), k_2 = \mu_1(\mu_1+\delta_1+\gamma),$
$k_3 = k_1 q\beta_H\theta N_H, k_4 = \alpha_4\beta_H\theta N_R(\mu_1+\delta_1), k_5 = k_1(\mu_1+\delta_1), k_6 = p\beta_H(1-\theta)N_H,$ (23)
$k_7 = \alpha_3\beta_R\theta N_R(\mu_1+\delta_1), k_8 = (\mu_1+\delta)k_1$

Substituting λ_F^* (19) into λ_M^* (22) we have

$$\lambda_M^* = \frac{k_6k_3\lambda_M^* + 2k_4\lambda_M^* + 2k_2\alpha_4\sigma\beta_R N_R + k_7k_3\lambda_M^* + k_2\alpha_3\sigma\beta_R N_R(k_5\lambda_M+k_2k_1)}{k_8k_3\lambda_M^* + k_4\lambda_M^* + k_2\alpha_4\sigma\beta_R N_R + k_2k_1} \quad (24)$$

$$\lambda_M^* = \frac{\lambda_M^*(k_6k_3+2k_4+k_7k_3+k_2k_5\alpha_3\sigma\beta_R N_R)+\sigma\beta_R N_R(\alpha_3k_2^2k_1+2k_2\alpha_4)}{\lambda_M^*(k_8k_3+k_4)+k_2\alpha_4\sigma\beta_R N_R+k_2k_1} \quad (25)$$

$$\lambda_M^*(\lambda_M^*(k_8k_3+k_4)+k_2\alpha_4\sigma\beta_R N_R+k_2k_1) = \lambda_M^*(k_6k_3+2k_4+k_7k_3+k_2k_5\alpha_3\sigma\beta_R N_R)+ \quad (26)$$
$$\sigma\beta_R N_R(\alpha_3k_2^2k_1+2k_2\alpha_4)$$

$$(\lambda_M^*)^2(k_8k_3+k_4)+\lambda_M^*k_2\alpha_4\sigma\beta_R N_R+\lambda_M^*k_2k_1-\lambda_M(k_6k_3+2k_4+k_7k_3+k_2k_5\alpha_3\sigma\beta_R N_R)- \quad (27)$$
$$\sigma\beta_R N_R(\alpha_3k_2^2k_1+2k_2\alpha_4)=0$$

$$(\lambda_M^*)^2(k_8k_3+k_4)+\lambda_M^*[k_2\alpha_4\sigma\beta_R N_R+k_2k_1-(k_6k_3+2k_4+k_7k_3+k_2k_5\alpha_3\sigma\beta_R N_R)]- \quad (28)$$
$$\sigma\beta_R N_R(\alpha_3k_2^2k_1+2k_2\alpha_4)=0$$

$$k_9(\lambda_M^*)^2+k_{10}\lambda_M^*+k_{11}=0 \quad (29)$$

Where

$$k_9 = (k_8 k_3 + k_4),$$
$$k_{10} = [k_2 \alpha_4 \sigma \beta_R N_R + k_2 k_1 - (k_6 k_3 + 2k_4 + k_7 k_3 + k_2 k_5 \alpha_3 \sigma \beta_R N_R)], \quad (30)$$
$$k_{11} = \sigma \beta_R N_R (\alpha_3 k_2^2 k_1 + 2k_2 \alpha_4$$

Thus, the positive endemic equilibria of the basic model (1) to (6) are obtained by solving for λ_M^* from the quadratic (29) and substituting the results (positive values of λ_M^* into the expressions in (16). Clearly, the coefficient k_9 of (29), is always positive, and k_{11} is positive (negative) if R_0 is less than (greater than) unity, respectively.

Asymptotic Global Stability of Endemic Equilibrium

We used the Centre Manifold theorem as described in (Castillo- Chavez and Songs, 2004), for Bifurcation analysis to show that our model equations (1) to (6) is globally asymptotically stable (GAS).

In order to apply this theorem, we first make the following change of variables. Let $S_1 = x_1, I_1 = x_{2,}, S_2 = x_3 = I_2 = x_4, R_1 = x_5, R_2 = x_6$, so that $N_H = x_1 + x_{2,} + x_3 + x_4$ and $N_R = x_5 + x_6$ further, using the vector notation, $X = (x_1, x_{2,}, x_3 x_4, x_5, x_6)^T$.

Then our model equations (1) to (6) can be written in the form $\frac{dx}{dt} = (f_1, f_{2,}, f_3, f_4, f_5, f_6)^T$, such that:

$$\frac{dx_1}{dt} = f_1 = \beta_H \theta N_H + \gamma x_2 - \frac{(px_4 + \alpha_3 x_6)x_1}{N_H} - \mu_1 x_1 \quad (31)$$

$$\frac{dx_2}{dt} = f_2 = \frac{(px_4 + \alpha_3 x_6)x_1}{N_H} - (\mu_1 + \delta_1 + \gamma)x_2 \quad (32)$$

$$\frac{dx_3}{dt} = f_3 = \beta_H (1-\theta) N_H + \gamma x_4 - \frac{(qx_2 + \alpha_4 x_6)x_3}{N_h} - \mu_1 x_3 \quad (33)$$

$$\frac{dx_4}{dt} = f_4 = \frac{(qx_2 + \alpha_4 x_6)x_3}{N_h} - (\mu_1 + \delta_1 + \gamma)x_4 \quad (34)$$

$$\frac{dx_5}{dt} = f_5 = \beta_R N_R - (\sigma + \mu_2 + \delta_2)x_5 \quad (35)$$

$$\frac{dx_6}{dt} = \sigma x_5 - (\mu_2 + \delta_2)x_6 \quad (36)$$

Now the Jacobian of the model equations (31) to (36) at disease free equilibrium which is;

$$J_{E_0} = \begin{bmatrix} -\mu_1 & \gamma & 0 & \frac{px_1}{N_H} & 0 & \frac{\alpha_3 x_1}{N_H} \\ 0 & -A_1 & 0 & \frac{px_1}{N_H} & 0 & \frac{\alpha_3 x_1}{N_H} \\ 0 & \frac{qx_3}{N_H} & -\mu_1 & 0 & 0 & \frac{\alpha_4 x_3}{N_H} \\ 0 & \frac{qx_3}{N_H} & 0 & -A_1 & 0 & \frac{\alpha_4 x_3}{N_H} \\ 0 & 0 & 0 & 0 & -A_{2v} & 0 \\ 0 & 0 & 0 & 0 & \sigma & -(\mu_2 + \delta_2) \end{bmatrix} \quad (37)$$

The basic reproductive number in terms of the new variables is $R_0 = \sqrt{\frac{px_1.qx_3.}{(N_H A_1)^2}}$

Consider the case where $R_0 = 1$, suppose that p is chosen as the Bifurcation parameter, since R_0 is often not convenient to use directly as bifurcation parameter, solving for p gives $R_0 = 1$, where $p^* = p$. Eigen-vectors of $J(_{E_0}) / p^* = p$

As in Garba et al (2008) we let v and w be the corresponding Right and Left eigen-vector associated with the zero eigen-value of the Jacobian (37) at $p^* = p$, denoted by J_{P^*}, chosen such that $vJ_{(E_0)} = 0$ and $J_{(E_0)} w = 0$,with $vw = 1$ where $v = (v_1, v_2, v_3, v_4, v_5, v_6)$, and $w = (w_1, w_2, w_3, w_4, w_5, w_6)$

$$J_{(E_0)}V = \begin{bmatrix} -\mu_1 & \gamma & 0 & \frac{px_1}{N_H} & 0 & \frac{\alpha_3 x_1}{N_H} \\ 0 & -A_1 & 0 & \frac{px_1}{N_H} & 0 & \frac{\alpha_3 x_1}{N_H} \\ 0 & \frac{qx_3}{N_H} & -\mu_1 & 0 & 0 & \frac{\alpha_4 x_3}{N_H} \\ 0 & \frac{qx_3}{N_H} & 0 & -A_1 & 0 & \frac{\alpha_4 x_3}{N_H} \\ 0 & 0 & 0 & 0 & -A_2 & 0 \\ 0 & 0 & 0 & 0 & \sigma & -(\mu_2 + \delta_2) \end{bmatrix} . \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = 0 \qquad (38)$$

To solve for V we reduce $J_{(E_0)}$, to upper triangular matrix using elementary row operation to have

$$\begin{bmatrix} -\mu_1 & \gamma & 0 & \frac{px_1}{N_H} & 0 & \frac{\alpha_3 x_1}{N_H} \\ 0 & -A_1 & 0 & \frac{px_1}{N_H} & 0 & \frac{\alpha_3 x_1}{N_H} \\ 0 & 0 & -\mu_1 & A_3 & 0 & A_4 \\ 0 & 0 & 0 & -A_1 + A_3 & 0 & A_4 \\ 0 & 0 & 0 & 0 & -A_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\mu_2 + \delta_2) \end{bmatrix} . \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = 0 \qquad (39)$$

where

$$A_1 = (\mu_1 + \delta_1 + \gamma),\ A_2 = (\sigma + \mu_2 + \delta_2),\ A_3 = \frac{px_3 px_1}{(N_H)^2 A_1},\ A_4 = \frac{qx_3 \alpha_3 x_1}{(N_H)^2 A_1} + \frac{\alpha_4 x_3}{N_H} \qquad (40)$$

Solving for v in equation (39) we have

$$v_1 = \frac{[(pA_4 + \alpha_1 x_1 (A_1 - A_2))\gamma + \alpha_3 A_4 + \alpha_3 (A_1 - A_3)]x_1}{N_H (A_1 - A_3)\mu_1} v_6 \qquad (41)$$

$$v_2 = \frac{p_1 x_1 A_4 + \alpha_3 x_1 (A_1 - A_3)}{A_1 - A_3} v_6 \qquad (42)$$

$$v_3 = \frac{A_3 A_4 - A_4 (A_1 - A_2)}{A_1 - A_3} v_6 \qquad (43)$$

$$v_4 = \frac{A_4}{A_1 - A_3} v_6 \qquad (44)$$

$$v_5 = 0 \quad (45)$$

$$v_6 = v_6 \quad (46)$$

Left eigenvalue $w.J(_{E_0})$

The left eigenvalue of (J_{E_0}) are transposes of the right eigenvectors of the transposed matrix $(J_{E_0})^T$, since their defining equation is equivalent to

$J_{(E_0)}{}^T w^T = \lambda w$, thus, $wJ_{(E_0)} = \lambda w$

that is

$$(J_{E_0})^T w^T = (w_1, w_{2,}, w_3, w_4, w_5, w_6)^T = \begin{bmatrix} -\mu_1 & 0 & 0 & 0 & 0 & 0 \\ \gamma & -A_1 & \frac{qx_3}{N_H} & \frac{qx_3}{N_H} & 0 & 0 \\ 0 & 0 & -\mu_1 & 0 & 0 & 0 \\ \frac{px_1}{N_H} & \frac{px_1}{N_H} & 0 & -A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_2 & 0 \\ \frac{\alpha_3 x_1}{N_H} & \frac{\alpha_3 x_1}{N_H} & \frac{\alpha_4 x_3}{N_H} & \frac{\alpha_4 x_3}{N_H} & 0 & -(\mu_2 + \delta_2) \end{bmatrix}.$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = 0 \quad (47)$$

To solve for W we reduce $J_{(E_0)}$, to lower triangular matrix using elementary row reduction to have

$$\begin{bmatrix} -\mu_1 & 0 & 0 & 0 & 0 & 0 \\ (A_5 + \gamma) & (-A_1 + A_3) & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu_1 & 0 & 0 & 0 \\ \frac{p_1 x_1^*}{N_h^*} & \frac{p_1 x_1^*}{N_h^*} & 0 & -A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_2 & 0 \\ \frac{\alpha_3 x_1}{N_H} & \frac{\alpha_3 x_1}{N_H} & \frac{\alpha_4 x_3}{N_H} & \frac{\alpha_4 x_3}{N_H} & 0_5 & (\mu_2 + \delta_2) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = 0. \quad (48)$$

where

$$A_5 = \frac{qx_3 R_3}{(N_H)^2 A_1} + R_2 \quad (49)$$

Solving for w in equation (48), we have

$$w_1 = w_1 \quad (50)$$

$$w_2 = \frac{A_5 + \gamma}{A_1 - A_3} w_1 \quad (51)$$

$$w_3 = 0 \quad (52)$$

$$w_4 = \frac{\big((A_2 - A_5) + (A_5 + \gamma)\big)px_1}{A_1 N_H (A_1 - A_5)} w_1 \qquad (53)$$

$$w_5 = 0 \qquad (54)$$

$$w_6 = \left(\frac{(A_1 + \gamma)}{N_H (A_1 - A_5)} x_1 \left(\frac{\alpha_4 N_H A_1 + \alpha_4 x_3 p}{N_H A_1} \right) w_1 + \frac{\alpha_4 x_3}{N_H} \right) \frac{1}{\mu_2 - \delta_{2\,1}} \qquad (55)$$

Computation of a and b

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0) \qquad (56)$$

For our model, we have $n = 6$, $k = 1,2,3,4$ representing the susceptible compartments, S_1, S_2, the dormant and active reservoir compartments R_1, R_2 therefore, $v_1 = v_3 = v_5 = v_6 = 0 \Rightarrow a_1 = a_3 = a_5 = a_6 = 0$. We therefore compute the associated non-zero partial derivatives of f at the DFE for $f_2 = f_4$

For f_2

$$f_2 = \frac{(px_4 + \alpha_3 x_6)x_1}{N_H} - (\mu_1 + \delta_1 + \gamma)x_2 \qquad (57)$$

$\frac{\partial^2 f_2}{\partial x_1 \partial x_j}(0,0) = 0$, for $j = 1,2,3,5$ $\quad \frac{\partial^2 f_2}{\partial x_1 \partial x_4}(0,0) = \frac{p}{N_H}$, $\quad \frac{\partial^2 f_2}{\partial x_1 \partial x_6}(0,0) = \frac{\alpha_3}{N_H}$

$\frac{\partial^2 f_2}{\partial x_2 \partial x_1}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_2 \partial x_2}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_2 \partial x_3}(0,0) = 0$,

$\frac{\partial^2 f_2}{\partial x_2 \partial x_4}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_2 \partial x_5}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_2 \partial x_6}(0,0) = 0$. $\frac{\partial^2 f_2}{\partial x_3 \partial x_1}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_3 \partial x_2}(0,0) = 0$,

$\frac{\partial^2 f_2}{\partial x_3 \partial x_3}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_3 \partial x_4}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_3 \partial x_5}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_3 \partial x_6}(0,0) = 0$. $\frac{\partial^2 f_2}{\partial x_4 \partial x_1}(0,0) = \frac{p}{N_H}$,

$\frac{\partial^2 f_2}{\partial x_4 \partial x_2}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_4 \partial x_3}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_4 \partial x_4}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_4 \partial x_5}(0,0) = 0$,

$\frac{\partial^2 f_2}{\partial x_4 \partial x_6}(0,0) = 0$. $\frac{\partial^2 f_2}{\partial x_5 \partial x_1}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_5 \partial x_2}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_5 \partial x_3}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_5 \partial x_4}(0,0) = 0$,

$\frac{\partial^2 f_2}{\partial x_5 \partial x_5}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_5 \partial x_6}(0,0) = 0$. $\frac{\partial^2 f_2}{\partial x_6 \partial x_1}(0,0) = \frac{\alpha_3}{N_H}$, $\frac{\partial^2 f_2}{\partial x_6 \partial x_2}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_6 \partial x_3}(0,0) = 0$,

$\frac{\partial^2 f_2}{\partial x_6 \partial x_4}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_6 \partial x_5}(0,0) = 0$, $\frac{\partial^2 f_2}{\partial x_6 \partial x_6}(0,0) = 0$

For f_4

$$f_4 = \frac{(qx_2 + \alpha_4 x_6)x_3}{N_H} - (\mu_1 + \delta_1 + \gamma)x_4 \qquad (58)$$

$\frac{\partial^2 f_4}{\partial x_1 \partial x_1}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_1 \partial x_2}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_1 \partial x_3}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_1 \partial x_4}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_1 \partial x_5}(0,0)=0$,
$\frac{\partial^2 f_4}{\partial x_1 \partial x_6}(0,0)=0$. $\frac{\partial^2 f_4}{\partial x_2 \partial x_1}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_2 \partial x_2}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_2 \partial x_3}(0,0)=\frac{q}{N_H}$, $\frac{\partial^2 f_4}{\partial x_2 \partial x_4}(0,0)=0$,
$\frac{\partial^2 f_4}{\partial x_2 \partial x_5}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_2 \partial x_6}(0,0)=0$. $\frac{\partial^2 f_4}{\partial x_3 \partial x_1}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_3 \partial x_2}(0,0)=\frac{q}{N_H}$, $\frac{\partial^2 f_4}{\partial x_3 \partial x_3}(0,0)=0$,
$\frac{\partial^2 f_4}{\partial x_3 \partial x_4}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_3 \partial x_5}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_3 \partial x_6}(0,0)=\frac{\alpha_4}{N_H}$. $\frac{\partial^2 f_4}{\partial x_4 \partial x_1}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_4 \partial x_2}(0,0)=0$,
$\frac{\partial^2 f_4}{\partial x_4 \partial x_3}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_4 \partial x_4}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_4 \partial x_5}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_4 \partial x_6}(0,0)=0$. $\frac{\partial^2 f_4}{\partial x_5 \partial x_1}(0,0)=0$,
$\frac{\partial^2 f_4}{\partial x_5 \partial x_2}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_5 \partial x_3}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_5 \partial x_4}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_5 \partial x_5}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_5 \partial x_6}(0,0)=0$.
$\frac{\partial^2 f_4}{\partial x_6 \partial x_1}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_6 \partial x_2}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_6 \partial x_3}(0,0)=\frac{\alpha_4}{N_H}$, $\frac{\partial^2 f_4}{\partial x_6 \partial x_4}(0,0)=0$, $\frac{\partial^2 f_4}{\partial x_6 \partial x_5}(0,0)=0$,
$\frac{\partial^2 f_4}{\partial x_6 \partial x_6}(0,0)=0$.

Therefore the associated non-zero partial derivative of f at DFE for the sign of a are given by;

$$\frac{\partial^2 f_2}{\partial x_1 \partial x_4}(0,0)=\frac{p}{N_H},\frac{\partial^2 f_2}{\partial x_1 \partial x_6}(0,0)=\frac{\alpha_3}{N_H},\frac{\partial^2 f_2}{\partial x_4 \partial x_1}(0,0)=\frac{p}{N_H},\frac{\partial^2 f_2}{\partial x_6 \partial x_1}(0,0)=\frac{\alpha_3}{N_H},$$
$$\frac{\partial^2 f_4}{\partial x_2 \partial x_3}(0,0)=\frac{q}{N_H},\frac{\partial^2 f_4}{\partial x_3 \partial x_2}(0,0)=\frac{q}{N_H},\frac{\partial^2 f_4}{\partial x_3 \partial x_6}(0,0)=\frac{\alpha_4}{N_H},\frac{\partial^2 f_4}{\partial x_6 \partial x_3}(0,0)=\frac{q}{N_h} \quad (59)$$

Substituting (59) into (56), we have

$$a_2 = 2v_2 w_1 w_4 \frac{p}{N_H} + 2v_2 w_1 w_6 \frac{\alpha_3}{N_H} \quad (60)$$

$$a_4 = 2v_4 w_2 w_3 \frac{q}{N_H} + 2v_4 w_3 w_6 \frac{\alpha_4}{N_H} \quad (61)$$

Simplifying, we have;

$$a_2 = 2v_2 w_1\left(w_4 \frac{p}{N_H} + w_6 \frac{\alpha_3}{N_H}\right) \qquad a_4 = 2v_4 w_3\left(w_2 \frac{q}{N_H} + w_6 \frac{\alpha_4}{N_H}\right)$$

$a = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$, but $a_1 = a_3 = a_5 = a_6 = 0$

Therefore, $a = 2v_2 w_1\left(w_4 \frac{p}{N_H} + w_6 \frac{\alpha_3}{N_H}\right) + 2v_4 w_3\left(w_2 \frac{q}{N_H} + w_6 \frac{\alpha_4}{N_H}\right)$ (62)

Clearly, if $w_1 > 0, w_2 > 0, w_3, > 0, w_4 > 0, w_5 > 0, w_6 > 0$, then, $a > 0$.

Otherwise, $w_1 < 0, w_2 < 0, w_3 < 0, w_4 < 0, w_5 < 0, w_6 < 0$ then, $a < 0$

Similarly, $b = \sum_{k,i,=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial x_\phi}(0,0)$ (63)

For $k = 1,3,5,6 \Rightarrow v_1 = v_3 = v_5 = v_6 = 0 \Rightarrow b_1 = b_3 = b_5 = b_6 = 0$. We therefore compute the associated non-zero partial derivatives of f at the DFE for $f_2 = f_4$

where $\phi = p^*$

$$f_2 = \frac{(px_4 + \alpha_3 x_6)x_1}{N_H} - (\mu_1 + \delta_1 + \gamma)x_2 \tag{64}$$

$$\frac{\partial^2 f_2}{\partial x_4 \partial x_\phi}(0,0) = \frac{x_1}{N_H}, \frac{\partial^2 f_2}{\partial x_1 \partial x_\phi}(0,0) = \frac{x_1}{N_H}, \frac{\partial^2 f_2}{\partial x_2 \partial x_\phi}(0,0) = 0, \frac{\partial^2 f_2}{\partial x_4 \partial x_\phi}(0,0) = 0,$$

$$\frac{\partial^2 f_2}{\partial x_5 \partial x_\phi}(0,0) = 0, \frac{\partial^2 f_2}{\partial x_6 \partial x_\phi}(0,0) = 0 .$$

$$f_4 = \frac{(qx_2 + \alpha_4 x_6)x_3}{N_H} - (\mu_1 + \delta_1 + \gamma)x_4 \tag{65}$$

$$\frac{\partial^2 f_4}{\partial x_2 \partial x_\phi}(0,0) = , \frac{\partial^2 f_4}{\partial x_1 \partial x_\phi}(0,0) = 0, \frac{\partial^2 f_4}{\partial x_3 \partial x_\phi}(0,0) = 0, \frac{\partial^2 f_4}{\partial x_4 \partial x_\phi}(0,0) = 0, \frac{\partial^2 f_4}{\partial x_5 \partial x_\phi}(0,0) = 0,$$

$$\frac{\partial^2 f_4}{\partial x_6 \partial x_\phi}(0,0) = 0,$$

Substituting the derivative of (64) and (65) into (63) we have b as;

$$b_2 = v_2 w_4 \frac{x_1}{N_H} + v_2 w_1 \frac{x_4}{N_H}, \tag{67}$$

$$b_4 = 0$$

$$b = v_2 w_4 \frac{x_1}{N_H} + v_2 w_1 \frac{x_4}{N_H} \tag{68}$$

$$b = v_2 \left(\frac{w_4 x_1}{N_H} + \frac{w_1 x_4}{N_H} \right) \tag{69}$$

$x_1 > 0, x_4 > 0, w_4 > 0, w_1 > 0$. Hence, $b > 0$

Thus we claim the following

Corollary 1

If $R_0 > 1$

The endemic equilibrium points E_1 is locally asymptotically stable for R_0 close to 1

The bifurcation at $p^* = 0$ is subcritical when $a > 0, b < 0$

Conclusion

A Mathematical Model was developed for the spread and control of Lassa Lever. Key to our analysis is the basic reproductive number (R_0), which is an important threshold for disease control. The disease free equilibrium (DFE) and the endemic equilibrium were obtained. In analyzing the endemic equilibrium states for stability we also adopted the method of (Castilo- Chavez and Song, 2004), which entails finding the Right and Left eigenvalues. The

analysis shows that the endemic equilibrium points E_1 is locally asymptotically stable for R_0 close to 1, and the bifurcation at $p^* = 0$ is subcritical when $a > 0$

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