# A COMPARISON OF THE RESPECTIVE ACCURACIES OF THE ADAMS-BASHFORTH AND MILNE'S METHODS FOR SOME INITIAL VALUE PROBLEMS (IVP)

Musa Bawa
Department Of Mathematics / Computer Science
Ibrahim Badamasi Babangida University, Lapai, Nigeria
Phone No: +234-805-825-9616

## Abstract

*This research work compared the effectiveness of the Adams-Bashforth method and the Milne's method as numerical methods of solving Ordinary Differential Equations. Approximate solutions were obtained for the first order initial value problems of the form;*

$$y' = f(x, y); \qquad y(x_0) = y_0$$

*and compared with the exact solution. It was discovered that the Milne's method performs better than the Adams-Bashforth method.*

## Introduction

Numerical methods are very helpful in obtaining approximate solutions to initial value problems at mesh points. This research work considers only first order initial value problems (IVP) of the form;

$$y'(x) = f(y(x)); \qquad y_0 = y(x_0) \qquad 1.1$$

Approximate solutions at the points $x_0, x_1, x_2, \ldots$ are generated, where the difference between any two successive x-values is the step size h; that is, $x_{n-1} - x_n = h$.

Two numerical methods used in solving Ordinary Differential Equations will be applied and their relative efficiency compared. These methods are the Adams- Bashforth method and the Milne's method. In each case, equation 1.1 is considered and solved using the two methods.

## Adams – Bashforth Method

This method uses the information at the past four stating values $y_1, y_2, y_3$ and $y_4$ to extrapolate the solution at the next point.

## Mathematical Expression for Adams- Bashforth Method

The first order ordinary differential equation as given in Butcher (2003) and Rhan (1984) is

$$y' = f(x, y) \qquad 2.1$$

Integrating equation (2.1) between $x = x_k$ and $x = x_{k+1}$, 2.2

we have

$$\int_{x_k}^{x_{k+1}} dy = \int_{x_k}^{x_{k+1}} f(x,y)dx \qquad 2.3$$

That is

$$y_{k+1} = y_k + \int_{x_k}^{x_{k+1}} f(x,y)dx \qquad 2.4$$

The integral on the right can be solved by approximating *f(x,y)* as a polynomial in *x.* This can be obtained by making it a fit through the past points $(x_{k-3}, y_{k-3})$, $(x_{k-2}, y_{k-2})$, $(x_{k-1}, y_{k-1})$ and $(x_k, y_k)$.

Suppose it is approximated using Newton's backward difference interpolating polynomial as applied in Girish (200), and carnahan etal (1969)

$$f(x,y) = f_k + u\nabla f_k + \frac{u(u+1)}{2}\nabla^2 f_k + \frac{u(u+1)(u+2)}{6}\nabla^3 f_k \qquad 2.5$$

where

$$u = \frac{x - x_k}{h} \qquad 2.6$$

and $\nabla f_k$, $\nabla^2 f_k$, $\nabla^3 f_k$ are backward differences at point $x = x_k$

Hence

$$Y_{k+1} = y_k + \int_{x_k}^{x_{k+1}} \left[ f_k + u\nabla f_k + \frac{u(u+1)}{2}\nabla^2 f_k + \frac{u(u+1)(u+2)}{6}\nabla^3 f_k \right] dx \qquad 2.7$$

From equation (2.6)

$dx = hdu$ 2.8

and at $x = x_k$, $u = 0$ 2.9

at $x = x_{k+1}$, $u = 1$ 2.10

Substituting equations (2.8) and (2.10) in equation (2.7) gives

$$Y_{k+1} = y_k + h\int_0^1 [\, f_k + u\nabla f_k + \frac{u(u+1)}{2}\nabla^2 f_k + \frac{u(u+1)(u+2)}{6}\nabla^3 f_k \,]\, du$$

$$Y_{k+1} = y_k + h\,[f_k + \frac{1}{2}\nabla f_k + \frac{5}{12}\nabla^2 f_k + \frac{3}{8}\nabla^3 f_k] \qquad 2.11$$

Expressing the backward differences in terms of function values and substituting in equation (2.11) gives

$$Y^p_{k+1} = y_k + \frac{h}{24}[-9f_{k-3} + 37f_{k-2} - 59f_{k-1} + 55f_k] \qquad 2.12$$

Equation (2.12) is called Adams Bashforth-formula of order four and is used as a predictor formula. The superscript P indicates the predicted value of $y_{k+1}$.

The corrector formula is developed similarly. Construct a Newton's backward difference interpolating polynomial passing through the points

$(x_{k-2}, y_{k-2})$, $(x_{k-1}, y_{k-1})$, $(x_x, y_k)$ and $(x_{k+1}, y^p{}_{k+1})$ as

$$f(x,y) = f_{k+1} + u\nabla f_{k+1} + \frac{u(u+1)}{2}\nabla^2 f_{k+1} + \frac{u(u+1)(u+2)}{6}\nabla^3 f_{k+1} \qquad 2.13$$

where $u = \frac{x - x_{k+1}}{h}$ 2.14

and $\nabla f_{k+1}$, $\nabla^2 f_{k+1}, \nabla^3 f_{k+1}$ are backward differences at point $x = x_{k+1}$

substituting equation (2.13) in equation (2.4) and integrating between $x = x_k$ and $x = x_{k+1}$ gives

$$Y_{k+1} = y_k + \int_{X_k}^{X_{k+1}} [\, f_{k+1} + u\nabla f_{k+1} + \frac{u(u+1)}{2}\nabla^2 f_{k+1} + \frac{u(u+1)(u+2)}{6}\nabla^3 f_{k+1}]\, dx. \qquad 2.15$$

From equation 2.14

$dx = hdu$ 2.16

and at $x = x_k$, $u = -1$ 2.17

at $x = x_{k+1}$, $u = 0$ 2.18

Substituting equation (2.16) to (2.18) in equation (2.15) gives

$$Y_{k+1} = y_k + h\int_1^0 [\, f_{k+1} + u\nabla f_{k+1} + \frac{u(u+1)}{2}\nabla^2 f_{k+1} + \frac{u(u+1)(u+2)}{6}\nabla^3 f_{k+1}]\, du$$

$$Y_{k+1} = y_k + h\,[\, f_{k+1} - \frac{1}{2}\nabla f_{k+1} - \frac{1}{12}\nabla^2 f_{k+1} - \frac{1}{24}\nabla^3 f_{k+1}] \qquad 2.19$$

Expressing the backward differences in terms of function values, and substituting in equation (2.19) gives

$$Y^{C}_{k+1} = y_k + \frac{h}{24}[f_{k-2} - 5f_{k-1} + 19f_k + 9f^{p}_{k+1}] \qquad 2.20$$

Equation (2.20) is called Adam-moulton formula and is used as a corrector formula. The superscript C indicates the corrected value of $y_{k+1}$

The equations (2.12) and (2.20) constitute the Adam-bashforth-moulton predictor- corrector method.

### Milne's Method

The Milne's method, like Adam – Bashforth method uses the information at past four solution points to extrapolate the solution at the next point. Therefore, in order to apply Milne's method, three more solution points, in addition to the starting solution point are computed.

### Mathematical Expression for Milne's Method.

From equation (2.1)

$$y' = f(x, y)$$

Integrating between $x = x_{k-3}$ and $x = x_{k+1}$, we have

$$y_{k+1} = y_{k+3} + \int_{x_{k-3}}^{x_{k+i}} f(x, y)dx \qquad 3.1$$

To solve the integral on the right approximate $f(x,y)$ as a polynomial in $x$. This is obtained by making it a fit through the past points $(x_{k-3}, y_{k-3})$, $(x_{k-2}, y_{k-2})$, $(x_{k-1}, y_{k-1})$ and $(x_k, y_k)$. suppose it is approximated using Newton's forward difference interpolating polynomial as in John and Kurtis (2004)

$$f(x,y) = f_{k-3} + u\Delta f_{k-3} + \frac{u(u-1)}{2}\Delta^2 f_{k-3} + \frac{u(u-1)(u-2)}{6}\Delta^3 f_{k-3} \qquad 3.2$$

where

$$u = \frac{x - x_{k-3}}{h} \qquad 3.3$$

and $\Delta f_{k-3}$, $\Delta^2 f_{k-3}$, $\Delta^3 f_{k-3}$ are forward differences at point $x = x_{k-3}$.

Therefore, equation (3.1) becomes

$$Y_{k+1} = y_{k+3} + \int_{x_{k-3}}^{x_{k+1}}[f_{k-3} + u\Delta f_{k-3} + \frac{u(u-1)}{2}\Delta^2 f_{k-3} + \frac{u(u-1)(u-2)}{6}\Delta^3 f_{k-3}]\, dx \qquad 3.4$$

From equation (3.3)

$$dx = hdu \qquad 3.5$$

$$\text{and at } x = x_{k-3},\ u = 0 \qquad 3.6$$

$$\text{at } x = x_{k+1},\ u = 4 \qquad 3.7$$

Substituting equation (3.5) to (3.7) in equation (3.4) gives

$$Y_{k+1} = y_{k+3} + h\int_0^4[f_{k-3} + u\Delta f_{k-3} + \frac{u(u-1)}{2}\Delta^2 f_{k-3} + \frac{u(u-1)(u-2)}{6}\Delta^3 f_{k-3}]\, du$$

$$Y_{k+1} = y_{k-3} + h[4f_{k-3} + 8\Delta f_{k-3} + \frac{20}{3}\Delta^2 f_{k-3} + + \frac{8}{3}\Delta^3 f_{k-3}] \qquad 3.8$$

Expressing the forward differences in terms of function values, and substituting in equation (3.8) gives

$Y^{p}_{k+1} = y_{k-3} + \frac{4h}{3} [2f_{k-2} - f_{k-1} + 2\ f_{k}]$ 3.9

Equation (3.9) is called Milne's formula of order four and is used as a predictor formula. The superscript P indicates the predicted value of $y_{k+1}$. To derive the corrector formula, construct a Newton's difference interpolating polynomial passing through the points $(x_{k-1}, y_{k-1})$, $(x_k, y_k)$ and $(x_{k+1}, y^{P}_{k+1})$ as:

$f(x,y) = f_{k+1} + u\Delta f_{k-1} + \frac{u(u-1)}{2}\Delta^{2} f_{k-1}$ 3.10

To evaluate the integral $\int_{x_{k-1}}^{x_{k+1}} f(x,y)dx$ in order to obtain the value of $y_{k+1}$ as $y_{k+1} = y_{k-1} + \int_{x_{k-1}}^{x_{k+1}} f(x,y)dx$ 3.11

By Simpson's $\frac{1}{3}$ rd rule for numerical integration

$\int_{x_{k-1}}^{x_{k+1}} f(x,y)dx = \frac{h}{3} [f_{k-1} + 4f_{k} + f^{P}_{k+1}]$ 3.12

Substituting equation (3.12) in equation (3.11) we have:

$Y^{c}_{k+1} = y_{k+1} + \frac{h}{3} [f_{k-1} + 4f_{k} + f^{P}_{k+1}]$ 3.13

Equation (3.13) is called Simpson's formula and is used as a corrector formula. The Superscript C indicates the corrected value of $y_{k+1}$. Hence equation (3.9) and (3.13) constitute the Milne – Simpson's predictor corrector method popularly known as Milne's method.

## Starting Values

The Adams – Bashforth method and Milne's method both require information at $y_1, y_2$ and $y_3$ to start. The first of these values is given by the initial condition. In the first – order initial value problems of the form.

$y' = f(x,y); \quad y(x_0) = y_0$

The other three starting values are obtained by the Runge- Kutta method

## Analysis and Results

In order to appreciate the differences between the two methods, we now turn the theoretical properties of the methods into computational reality. Hence, this section presents examples to illustrate the use of the two methods.

## Example 1.1

Using both methods to solve $y' = y - x;\ y(0) = 2$ on the interval [0,1] with h=0.1

Solution:

Here $f(x,y) = y-x$, $x_0 = 0$ and $y_0 = 2$, the three additional starting values are obtained using Runge-Kutta method as $y_1 = 2.2051708$, $y_2 = 2.4214026$ and $y_3 = 2.6498585$

Hence

$y_1^1 = y_1 - x_1 = 2.1051708$

$y_2^1 = y_2 - x_2 = 2.2214026$

$y_3^1 = y_3 - x_3 = 2.3498585$

Then using equation (2.12) and (2.20), we compute and generate the table below for Adams – Bashforth method and equations (3.9) and (3.13) for Milne's method.

Table 1.1 Results generated using Adams-Bashforth method for the problem $y^1 = y - x;\ y(0) = 2$.

| $X_n$ | h=0.1 | | Exact Solution |
|---|---|---|---|
| | $Py_n$ | $Y_n$ | (x) = $e^x + x + 1$ |
| 0.0 | - | 2.0000000 | 2.0000000 |
| 0.1 | - | 2.2051708 | 2.2051709 |
| 0.2 | - | 2.4214026 | 2.4214028 |
| 0.3 | - | 2.6498585 | 2.6498588 |
| 0.4 | 2.8918201 | 2.8918245 | 2.8918247 |
| 0.5 | 3. 1487164 | 3.1487213 | 3.1487213 |
| 0.6 | 3. 4221137 | 3.4221191 | 3.4221188 |
| 0.7 | 3. 7137473 | 3.7137533 | 3.7137527 |
| 0.8 | 4. 0255352 | 4.0255418 | 4.0255409 |
| 0.9 | 4. 3595971 | 4.3596044 | 4.3596031 |
| 1.0 | 4. 7182756 | 4.7182836 | 4.7182818 |

Table 1.2 Results generated using Milne's method for the problem $y' = y - x;\quad y(0) = 2$

| $X_n$ | h=0.1 | | Exact Solution |
|---|---|---|---|
| | $Py_n$ | $Y_n$ | (x) = $e^x + x + 1$ |
| 0.0 | - | 2.0000000 | 2.0000000 |
| 0.1 | - | 2.2051708 | 2.2051709 |
| 0.2 | - | 2.4214026 | 2.4214028 |
| 0.3 | - | 2.6498585 | 2.6498588 |
| 0.4 | 2.8918208 | 2.8918245 | 2.8918247 |
| 0.5 | 3. 1487169 | 3.1487209 | 3.1487213 |
| 0.6 | 3. 4221138 | 3.4221186 | 3.4221188 |
| 0.7 | 3. 7137472 | 3.7137524 | 3.7137527 |
| 0.8 | 4. 0255349 | 4.0255407 | 4.0255409 |
| 0.9 | 4. 3595964 | 4.3596027 | 4.3596031 |
| 1.0 | 4. 7182745 | 4.7182815 | 4.7182818 |

Example 1.2
Using both methods to solve $y^1 = y^2 + 1;\quad y(0) = 0,$ on the interval [0, 1] with h=0.1

Solution:
$f(x,y) = y^2 + 1$, *with* $x_0 = 0$, $y_0 = 0$. The three starting values are generated to be $y_1 = 0.1003346$, $y_2 = 0.2027099$, *and* $y_3 = 0.\ 3093360$.
Thus

$y_0^1 = (y_0)^2 + 1 = 1$

$y_1^1 = (y_1)^2 + 1 = 1.0100670$

$y_2^1 = (y_2)^2 + 1 = 1.0410913$

$y_3^1 = (y_2)^2 + 1 = 1.0956888$

Using equations (2.12) and (2.20), we compute and generate the table below for Adams Bashforth method and equations (3.9) and (3.13) for Milne's method.

Table 1.3: Results generated using Adams-Bashforth method for the problem $y^1 = y^2 + 1; y(0) = 0$

| $X_n$ | h=0.1 | | Exact Solution |
|---|---|---|---|
| | $Py_n$ | $Y_n$ | y(x) = tan x |
| 0.0 | - | 0.0000000 | 0.0000000 |
| 0.1 | - | 0.1003346 | 0.1003347 |
| 0.2 | - | 0.2027099 | 0.2027100 |
| 0.3 | - | 0.3093360 | 0.3093363 |
| 0.4 | 0.4227151 | 0.4227981 | 0.4227932 |
| 0.5 | 0.5461974 | 0.5463449 | 0.5463025 |
| 0.6 | 0.6839784 | 0.6841611 | 0.6841368 |
| 0.7 | 0.8420274 | 0.8423349 | 0.8422884 |
| 0.8 | 1.0291713 | 1.0297142 | 1.0296386 |
| 0.9 | 1.2592473 | 1.2602880 | 1.2601582 |
| 1.0 | 1.5554514 | 1.5576256 | 1.5574077 |

Table 1.4: Results generated using Milne's method for the problem $y^1 = y^2 + 1; y(0) = 0$

| $X_n$ | h=0.1 | | Exact Solution |
|---|---|---|---|
| | $Py_n$ | $Y_n$ | (x) = $e^x + x + 1$ |
| 0.0 | - | 0.0000000 | 0.0000000 |
| 0.1 | - | 0.1003346 | 0.1003347 |
| 0.2 | - | 0.2027099 | 0.2027100 |
| 0.3 | - | 0.3093360 | 0.3093363 |
| 0.4 | 0.4227227 | 0.4227946 | 0.4227932 |
| 0.5 | 0.5462019 | 0.5463042 | 0.5463025 |
| 0.6 | 0.6839791 | 0.6841405 | 0.6841368 |
| 0.7 | 0.8420238 | 0.8422924 | 0.8422884 |
| 0.8 | 1.0291628 | 1.0296421 | 1.0296368 |
| 0.9 | 1.2592330 | 1.2601516 | 1.2601582 |
| 1.0 | 1.5554357 | 1.5573578 | 1.5574077 |

Table 1.5: Comparison of the performances of Adams-Bashforth, Milne's and exact solution for the problem $y^1 = y - x; y(0) = 2, h = 0.1$

| $X_n$ | Y for ABM | Y for MM | Actual Y | Absolute error for ABM | Absolute error for MM |
|---|---|---|---|---|---|
| 0.0 | 2.0000000 | 2.0000000 | 2.0000000 | 0.0000000 | 0.0000000 |
| 0.1 | 2.2051708 | 2.2051708 | 2.2051709 | 0.0000001 | 0.0000001 |
| 0.2 | 2.4214026 | 2.4214026 | 2.4214028 | 0.0000002 | 0.0000002 |
| 0.3 | 2.6498585 | 2.6498585 | 2.6498588 | 0.0000003 | 0.0000003 |
| 0.4 | 2.8918245 | 2.8918245 | 2.8918247 | 0.0000002 | 0.0000002 |
| 0.5 | 3.1484213 | 3.1487209 | 3.1487213 | 0.0000000 | 0.0000004 |
| 0.6 | 3.4221191 | 3.4221186 | 3.4221188 | 0.0000003 | 0.0000002 |
| 0.7 | 3.7137533 | 3.7137524 | 3.7137527 | 0.0000006 | 0.0000003 |
| 0.8 | 4.0255418 | 4.0255407 | 4.0255409 | 0.0000009 | 0.0000002 |
| 0.9 | 4.3596044 | 4.3596027 | 4.3596031 | 0.0000013 | 0.0000004 |
| 1.0 | 4.7182836 | 4.7182815 | 4.7182818 | 0.0000018 | 0.0000003 |

Key
ABM = Adams-Bashforth Method
MM = Milne's Method

Table 1.6: Comparison of the performances of Adams-Bashforth, Milne's and exact solution for the problem $y' = y^2 + 1; \quad y(0) = 0$

| $X_n$ | Y for ABM | Y for MM | Actual Y | Absolute error for ABM | Absolute error for MM |
|---|---|---|---|---|---|
| 0.0 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.1 | 0.1003346 | 0.1003346 | 0.1003347 | 0.0000001 | 0.0000001 |
| 0.2 | 0.2027099 | 0.2027099 | 0.2027100 | 0.0000001 | 0.0000001 |
| 0.3 | 0.3093360 | 0.3093360 | 0.3093363 | 0.0000003 | 0.0000003 |
| 0.4 | 0.4227981 | 0.4227946 | 0.4227932 | 0.0000049 | 0.0000014 |
| 0.5 | 0.5463449 | 0.54630425 | 0.5463025 | 0.0000424 | 0.0000017 |
| 0.6 | 0.6841611 | 0.6841405 | 0.6841368 | 0.0000243 | 0.0000037 |
| 0.7 | 0.8423349 | 0.8422924 | 0.8422884 | 0.0000465 | 0.0000040 |
| 0.8 | 1.0297142 | 1.0296421 | 1.0296386 | 0.0000756 | 0.0000035 |
| 0.9 | 1.2602880 | 1.2601516 | 1.2601582 | 0.0001298 | 0.0000066 |
| 1.0 | 1.5576256 | 1.5573578 | 1.5574077 | 0.0002179 | 0.0000499 |

Key
ABM = Adams-Bashforth's Method
MM = Milne's Method

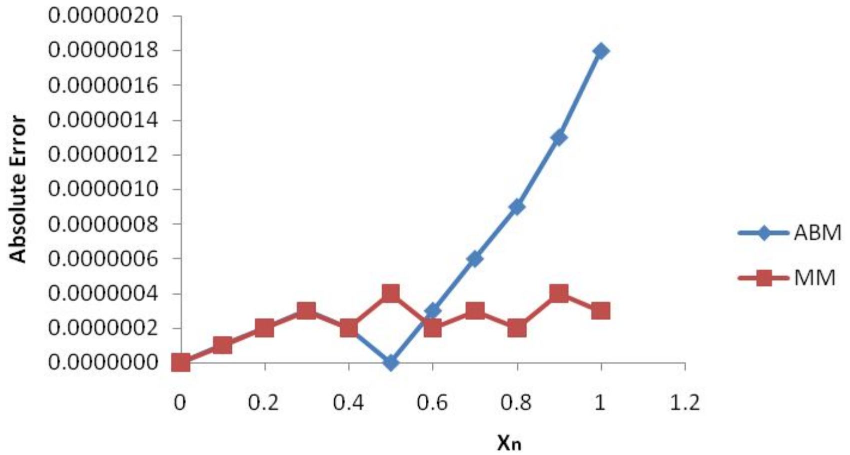


Fig .1.1: Graphical Representation of the Absolute Errors for y' = y - x

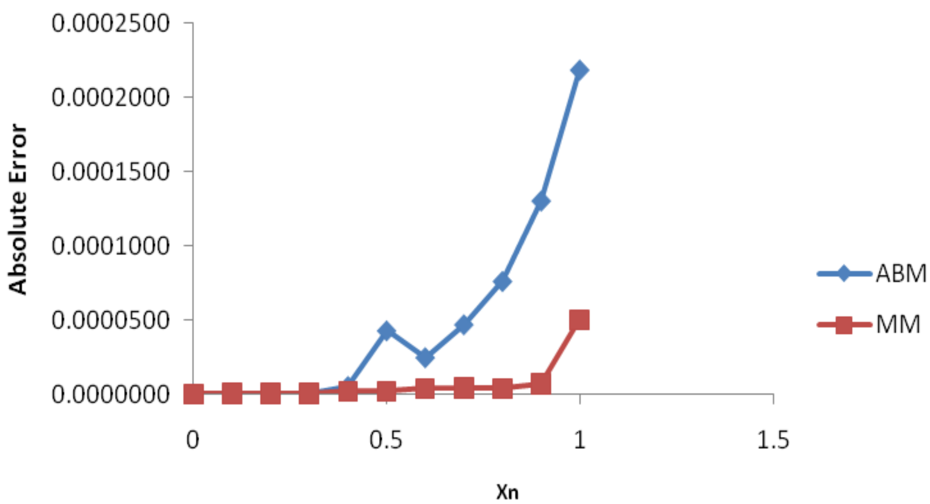


Fig 1.2 : Graphical Representation of the Absolute Errors for y = y² + 1

Looking at the tables and graphical representations of the absolute errors of the two methods, it is clear that as the $x_n$ values approaches one, the absolute error margin gets bigger. For table 1.5 and graph 1.1, the percentage error for ABM at $x_n$ = 1.0 is 0.000038% while that of MM is 0.000006%. similarly, from table 1.6 and graph 1.2, the percentage error for ABM at $x_n$ =1.0 is 0.0139912% while that of MM is 0.0032048%.

## Conclusion

From the tables of results, graphs and percentage error of each of the methods employed, the reliability of the Milne's method over Adams Bashforth's method is self evident. The Milne's method will be credited for its higher accuracy.

## References

Butcher, J. C. (2003). *Numerical methods for ordinary differential equations*. John Wiley.

Carnahan, B., Luther, H. A. & Wilkies, J. O. (1969). *Applied numerical methods*. New York: John Wiley and Sons.

Girish, N. (2009). *Numerical methods (A programming approach).* New Delhi: S. K. Kataria and Sons.

John, H. M. & Kurtis, K. F. ( 2004). *Numerical methods using matlab*. New Jersey: USA.Prentice-Hall Inc.

Rhan, D. M. (1984). *Penalty and barrier functions" numerical methods for constrained optimization.* London: Academic Press Inc.