VARIATIONAL ITERATION METHOD FOR THE SOLUTION OF LINEAR WAVE EQUATION

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Abstract
In this study, He's variational Iteration method is presented as an alternative method of solving the linear wave equation. Some numerical examples are selected to illustrate the effectiveness and accuracy of the method. It is observed that the method is very efficient and rapidly converges as exact solutions are obtained after few iterations.

Keywords: variational iteration method, Lagrange multiplier, differential equations, convergence.

Introduction

The variational iteration method was developed by He (1999, 2000, 2005, 2006a). In recent years a great deal of attention has been devoted to the study of the method. The reliability of the method and the reduction in the size of the computational domain give this method a wide applicability (Gorji-Bandpy et. al. (2007), Ganjavi et. al. (2008) and Onur et. al. (2010)). The present technique requires no restrictive assumptions that are used to handle nonlinear terms. The variational iteration method does not require specific transformation for terms in the equation as required by other techniques (Hussian and Majid (2010)). In this paper, variational iteration method is implemented for finding the exact solutions to linear wave equation.

He's Variational Iteration Method

To illustrate the basic concept of variational iteration method (VIM), we consider the following differential equation.

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \qquad 2.1$$

Where L is a linear operator, R is the remaining linear operator, N a nonlinear operator and $g(x,t)$ a inhomogeneous term. By the variational iteration method, we can construct a functional as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(t,s)\left(Lu_n(x,s)+Ru_n(x,s)+N\tilde{u}_n(x,s)-g(x,s)\right)ds \qquad 2.2$$

$$n \geq 0$$

Where $\lambda(t,s)$ is a general Lagrange multiplier which can be identified optimally via the variational theory (He, (2006b)). The function $\tilde{u}_n$ is a restrictive variation i.e. $\delta\tilde{u}_n = 0$. Therefore we first determine the Lagrange multiplier λ, that can be identified optimally via integration by parts. The successive approximation $u_{n+1}(x,t);\ n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective $u_0(x,t)$. The zeroth approximation $u_0(x,t)$ may be selected by any function that satisfies at least two of the prescribed boundary condition (Saeed et. al.(2009)). With λ determined, then several approximation $u_{n+1}(x,t)$ follow immediately. Consequently, the exact solution may be obtained by using $u(x,t)=\lim\limits_{n\to\infty} u_n(x,t)$

Application of Variational Iteration Method
Considering the linear wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t) \tag{3.1}$$

Equation (3.1) has a wide range of applications in the fields of science and engineering. By the Variational iteration method, a correct functional for eqn. (3.1) can be constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)\left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - c^2 \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2} - g(x,s)\right)ds \qquad n \geq 0 \tag{3.2}$$

Its stationary conditions can be obtained as follows

$$\frac{\partial^2 \lambda(t,s)}{\partial s^2} = 0$$

$$1 - \left.\frac{\partial \lambda(t,s)}{\partial s}\right|_{t=s} = 0$$

$$\lambda(t,s)\big|_{t=s} = 0$$

The Lagrange multiplier, therefore can be identified as

$$\lambda(t,s) = s - t$$

As a result, the following iteration formula is obtained.

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - c^2 \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2} - g(x,s)\right)ds \tag{3.3}$$

Some Numerical Examples

1. Consider the homogeneous one dimensional wave equation

$$u_{tt} - u_{xx} = 0 \tag{3.4}$$

With conditions $u(0,t) = t^2$, $u(x,0) = x^2$ and $u_t(x,0) = 6x$

The following iteration formular is obtain

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2}\right)ds \tag{3.5}$$

The initial approximation is taken as

$$u_0(x,t) = x^2 + 6xt$$

Using equation (3.3) equation (3.5) yields the following

$$u_1(x,t) = x^2 + 6xt + t^2$$

$$u_2(x,t) = x^2 + 6xt + t^2$$

$$u_3(x,t) = x^2 + 6xt + t^2$$

$$u_n(x,t) = x^2 + 6xt + t^2$$

$u(x,t) = \underset{n\to\infty}{Lim} u_n(x,t)$, hence $u(x,t) = x^2 + 6xt + t^2$

2. Consider inhomogeneous one dimensional wave equation

$$u_{tt} - u_{xx} = t^7 \qquad 3.6$$

With initial conditions $u(x,0) = 2x + \sin x$, $u_t(x,0) = 2x$

The following iteration formular is obtained

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2} - s^7\right)ds \qquad 3.7$$

The initial approximation is taken as

$$u_0(x,t) = 2x + \sin x + 2xt$$

Using the above variation iteration formular equation (3.7) yields

$$u_1(x,t) = 2x + 2xt + \sin x\left(1 - \frac{t^2}{2!}\right) + \frac{t^9}{72}$$

$$u_2(x,t) = 2x + 2xt + \sin x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!}\right) + \frac{t^9}{72}$$

$$u_3(x,t) = 2x + 2xt + \sin x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right) + \frac{t^9}{72}$$

$$u_n(x,t) = 2x + 2xt + \sin x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots\ldots + (-1)^n \frac{t^{2n}}{(2n)!}\right) + \frac{t^9}{72}$$

$u(x,t) = \underset{n\to\infty}{Lim} u_n(x,t)$, hence $u(x,t) = 2x + 2xt + \sin x \cos t + \dfrac{t^9}{72}$

The result obtained is same as obtained by Yehuda and Jacob (2005)

3. Consider inhomogeneous one dimensional wave equation

$$u_{tt} - 9u_{xx} = e^x - e^{-x} \qquad 3.8$$

With initial conditions $u(x,0) = x$, $u_t(x,0) = \sin x$

The following iteration formular is obtain

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - 9\frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2} - e^x + e^{-x}\right)ds \qquad 3.9$$

The initial approximation is taken as

$$u_0(x,t) = x + t\sin x$$

Using the above variation iteration formular equation (3.9) yields

$$u_1(x,t) = x + \sin x\left(t - \frac{3t^3}{2}\right) + e^x\left(\frac{t^2}{2}\right) - e^{-x}\left(\frac{t^2}{2}\right)$$

$$u_2(x,t) = x + \sin x\left(t - \frac{3t^3}{2} + \frac{27t^5}{40}\right) + e^x\left(\frac{t^2}{2} + \frac{3t^4}{8}\right) - e^{-x}\left(\frac{t^2}{2} + \frac{3t^4}{8}\right)$$

$$u_3(x,t) = x + \sin x\left(t - \frac{3t^3}{2} + \frac{27t^5}{40} - \frac{81t^7}{560}\right) + e^x\left(\frac{t^2}{2} + \frac{3t^4}{8} + \frac{9t^6}{80}\right) - e^{-x}\left(\frac{t^2}{2} + \frac{3t^4}{8} + \frac{9t^6}{80}\right)$$

$$u_n(x,t) = x + \sin x\left(t - \frac{3t^3}{2} + \frac{27t^5}{40} - \frac{81t^7}{560} + \ldots\right) + e^x\left(\frac{t^2}{2} + \frac{3t^4}{8} + \frac{9t^6}{80} + ..\right) - e^{-x}\left(\frac{t^2}{2} + \frac{3t^4}{8} + \frac{9t^6}{80} + ..\right)$$

$u(x,t) = \underset{n\to\infty}{Lim} u_n(x,t)$, hence $u(x,t) = x + \dfrac{\sin x \sin 3t}{3} + \dfrac{1}{9}\left(e^x - e^{-x}\right)(\cosh 3t - 1)$

4. Consider inhomogeneous one dimensional wave equation

$$u_{tt} - 4u_{xx} = 6t \qquad 3.10$$

With conditions, $u(x,0) = x$ and $u_t(x,0) = 0$

The following iteration formular is obtain

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - 4\frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2} - 6s\right)ds \qquad 3.11$$

The initial approximation is taken as

$$u_0(x,t) = x$$

Using the above variation iteration formular equation (3.11) yields

$$u_1(x,t) = x + t^3$$
$$u_2(x,t) = x + t^3$$
$$u_3(x,t) = x + t^3$$
$$u_n(x,t) = x + t^3$$

$u(x,t) = \underset{n\to\infty}{Lim} u_n(x,t)$, hence $u(x,t) = x + t^3$

5. Consider inhomogeneous one dimensional wave equation

$$u_{tt} - u_{xx} = xt \qquad 3.12$$

With initial conditions $u(x,0) = 0$, $u_t(x,0) = e^x$

The following iteration formular is obtained

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2} - xs\right)ds \qquad 3.13$$

The initial approximation is taken as

$u_0(x,t) = t\,e^x$

Using the above variation iteration formular equation (3.13) yields

$$u_1(x,t) = e^x\left(t + \frac{t^3}{3!}\right) + \frac{1}{6}xt^3$$

$$u_2(x,t) = e^x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!}\right) + \frac{1}{6}xt^3$$

$$u_3(x,t) = e^x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!}\right) + \frac{1}{6}xt^3$$

$$u_n(x,t) = e^x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + ... + \frac{t^{2n+1}}{(2n+1)!}\right) + \frac{1}{6}xt^3$$

$u(x,t) = \underset{n\to\infty}{Lim}\, u_n(x,t)$, hence $u(x,t) = e^x \sinh t + \frac{1}{6}xt^3$

Conclusion

The variational iteration method is employed in this work to obtain exact solutions to the linear wave equation. The variational iteration method reduces the size of calculations and do not require large computer memory and discretization of variable t as the exact solutions are obtained by fewer iterations. The initial approximations can be arbitrary chosen with unknown constants. It can be concluded that the variational iteration method is very powerful tool for solving linear and nonlinear initial value problem (IVPs). For computations in this paper, maple package was employed.

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