ON SOME CHARACTERIZATION OF INTEGRABILITY OF FUZZY SET

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Abstract
In this article, we investigate the concept of integrability of fuzzy sets and show that some established classical theorems of integral arevalid for the fuzzy integral. Some important and useful theorems of this integral are proved.

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Introduction
Fuzzy logic is a superset of convextional (Boolean) logic that has extented to handle the concept of partial truth, that is, true values between completely true and completely false [Zadeh (1965)] uses the interval value between 0 (false) and 1 (Truth) to describe any human reasoning.

This logic requires knowledge in order to reason and this knowlegde which is provided by an expert who knows the process is stored in the Fuzzy System [Zimmerrnann (1987)].

It is used in artifitial inteligence that deals with the reasoning. These are used in applications where process data cannot be represented in binary form.

It is being applied in knowlegde based automatic decision making [Zimmerrnann (1991)], forecast evaluation [Lee (2005)] and processes control engineering [Sugemo, (1985)]. A concept of a fuzzy integral was defined by [Sugeno, (1985)].

Ralescu and Adams (1980) define the fuzzy integral of a positive, measurable function, with respect to a fuzzy measure and show that the monotone convergence theorem and Fatou's lemma coincides with fuzzy integral. In particular, they estabished that convergence theorem is in a way stronger than the Lebesgue-dominated convergence theorem. Our main goal in this paper is, therefore, to estabilsh some basic theorem on integral of fuzzy sets using the fuzzy measures which are applicable in various areas of life [Rudin, (1974)].

1.1 Basic Operation of Fuzzy set
The following operations can be found in all standard texts of fuzzy sets theory and in literatures like [Zimmermann, (1991), Omolehin, et. al., (2005) and Rauf, at. al. (2012)].

1.1.1 Degree of Membership function:
The membership degree of any fuzzy sets, say K, denoted by $\mu_k(x)$ is $0 \leq \mu_k(x) \leq 1$.

1.1.2 Union:
The Union of fuzzy sets K and R called K union R is

$\mu_{A\cup B}(x) = \max\{\mu_A, \mu_B\}$, $\forall\, x \in X$

1.1.3 Intersection:

The Intersection of two fuzzy sets K and R is $\mu_{A\cap B}(x) = \min\{\mu_A, \mu_B\}$, $\forall\, x \in X$.

1.1.4 Complement:

The complement of aset K is denoted by $\bar{K}$ with a membership degree of
$\mu_{\bar{A}}(x) = 1 - \mu_A(x),\ x \in X$

1.1.5 Subset:

Let K and R be two fuzzy sets, then K is a subset of R if K and
$K \neq R,\ \mu_K(x) \neq \mu_R(x)$ such that $\mu_K(x) \leq \mu_R(x), \forall\, x \in X$.
Conventionally, we write $K \subset R$. Futhermore, K is a proper set of R if
$\mu_K(x) < \mu_R(x), \forall\, x \in X$. Implying that $K \subset R$ if and only if
$K \sqsubseteq R \text{ and } K \neq R$.

1.1.6 Equivalent:

K is equivalent to R if their degrees of membership are the same, that is $\{\mu_K(x) = \mu_R(x),\ x \in X\}$.

1.1.7 α-cut of Fuzzy set:

α-cut of a fuzzy set is the set of all points $x \in X$ such that every membership function $\mu(x) \geq \alpha$. See [Buckley and Eslami, (2002), Kandel, (1986), Lee and Lee (2001)] for further information.

2. Statement of Results:

This section discuss some properties of integration of fuzzyfing function or set in a non-fuzzy interval. However, it is also possible to integrate non-fuzzy function or set in fuzzy interval $[K, R]$ where the boundaries are obtained by two fuzzy sets K and R. Throughout this paper, μ is finite, fuzzy and subadditive measure. Also, fuzzy almost everywhere implies fuzzy in measure.

Theorem 2.1

Let K and R be integrable Fuzzy sets and let U and V be any real numbers.
Then for $x \in [a, b] \in \mathbb{R}$ and μ is a measure, the following assertions are valid:

(1) If $uK + vR$ is an integrable fuzzy set, then
$\int_a^b (uK + vR)d\mu(x) = u\int_a^b K\, d\mu(x) + v\int_a^b R\, d\mu(x)$.

(2) Suppose $K \geq 0$ almost everywhere in the universal set X,
then $\int_a^b K d\mu(x) \geq 0$.

(3) Assume $K \geq R$ almost everywhere in X, then,
$\int_a^b K d\mu(x) \geq \int_a^b R d\mu(x)$.

(4) Let $|K|$ is a fuzzy set and $|\int_a^b K d\mu(x)| \leq \int_a^b |K| d\mu(x)$.

(5) $\int_a^b |K + R| d\mu(x) \leq \int_a^b |K| d\mu(x) + \int_a^b |R| d\mu(x)$.

(6) If $A \subseteq K \subseteq B$ almost everywhere on a set $E \subset X$ with $\mu(E) < \infty$, then $A\mu(E) \leq \int_a^b K\, d\mu \leq B\mu(E)$.

(7) Suppose $A \geq 0$ almost everywhere in X and K and R are fuzzy sets with $K \subseteq R$, we have $\int_K\ Ad\mu(x) \leq \int_R\ Ad\mu(x)$.

(8) If $K = \bigcup_{j=1}^{\infty} K_j$ and $K_I \cap K_J = \emptyset$ and $K_{j's}$ are fuzzy sets with finite measure, then $\int_K\ Rd\mu(x) = \sum_{j=1}^{\infty} K_j Rd\mu(x)$.

Proof of (1):

Let $x \in [a,b] \in \mathbb{R}$, then the integral of fuzzy set K on $[a,b]$ is $\int_a^b Kdx = \{\left(\int_a^b K_a^{(-)}(x)dx + \int_a^b K_a^{(+)}(x)dx,\ \alpha\right) : \alpha \in [0,1]\}$ where $K_a^{(+)}$ and $K_a^{(-)}$ are α-cut function of the fuzzy value for fuzzyfing function $K(x)$ while $(-)$ and $(+)$ represent enumeration in the fuzzy sey. Hence, $\int_a^b (uK + vR)d\mu(x) = \lim_{\mu \to 0} \sum_{j=1}^{n} \left(uK(\xi_j) + vR(\xi_j)\right)\Delta_j(x)$, where ξ is the upper limit of integration. Thus,

$$\lim_{\mu \to 0} \sum_{j=1}^{n} \left(uK(\xi_j) + vR(\xi_j)\right)\Delta_j(x) = u \lim_{\mu \to 0} \sum_{j=1}^{n} K(\xi_j)\Delta_j(x) + v \lim_{\mu \to 0} \sum_{j=1}^{n} R(\xi_j)\Delta_j(x)$$

$$= u \int_a^b K\, d\mu(x) + v \int_a^b R\, d\mu(x)$$

Proof of (2):

Let K be an integral fuzzy set defined on $[a,b] \in \mathbb{R}$ and since $K \geq 0$, we have the membership degree of K in the interval $0 \leq \mu_K(x) \leq 1$ and $\int_a^b Kd\mu(x) \geq 0$.

Proof of (3):

From proposition (2), we have $\int_a^b Kd\mu(x) \geq 0$, let $K \geq R$ then the degree of membership function of R can not be more than that of K. Hence, the area that will be covered by fuzzyfing set K in a non fuzzifying interval will be more or equal the area of R. Therefore, $\int_a^b Kd\mu(x) \geq \int_a^b Rd\mu(x)$.

Proof of (4):

Assume $|K|$ is a fuzzy set, then $\mu_{|K|}$ maps the elements in the universal set X to the set $\{0,1\}$, so also $|\int_a^b Kd\mu(x)|$ and by the principle of absolute value and since every subinterval of interval $(0,1)$ contains both rational and irrational values on the interval then $|\int_a^b Kd\mu(x)| \leq \int_a^b |K| d\mu(x)$.

Proof of (5):

Since $\mu_K > 0$ then $\mu_{|K|} > 0$, we can say that $\mu_K \subseteq \mu_{|K|}$, Also, if $\mu_R > 0$ then $\mu_{|R|} > 0$, impliying that $\mu_R \subseteq \mu_{|R|}$. Therefore, $\mu_{K+R} \subseteq \mu_{|K+R|} \subseteq \mu_{|K|} + \mu_{|R|}$ Hence, $\int_a^b |K+R| d\mu(x) \leq \int_a^b |K| d\mu(x) + \int_a^b |R| d\mu(x)$.

Proof of (6):

If $A \subseteq K \subseteq B$ almost everywhere on the universal set Xexcept at point zero on the set $E \subset X$ with $\mu(E) < 1$. Since $\int_a^b K d\mu(x)$ is positive and $A \subseteq K$ then $A\mu(E) \leq \int_a^b K d\mu(x)$. Similarly, $K \subseteq B$ implying $\int_a^b K d\mu(x) \leq B\mu(E)$. Therefore, $A\mu(E) \leq \int_a^b K d\mu(x) \leq B\mu(E)$.

Proof of (7):

Suppose A is a set (either fuzzyfing or not) with a fuzzyfing sets K and R. If $K \subseteq R$ then the interval in R can not less than that of K except if $R \subseteq K$ when equality occurs and hence $\int_K A d\mu(x) \leq \int_R A d\mu(x)$.

Proof of (8):

Assume $K_{j's}$ are fuzzy sets with finite measure and if $K_I \cap K_J = \emptyset$ then

$$\int_K R d\mu(x) = \int_{\bigcup_{j=1}^{\infty} K_J} R d\mu(x) = \sum_{j=1}^{\infty} K_J R d\mu(x).$$

Since $K = \bigcup_{J=1}^{\infty} K_J$ and hence the prove of the theorem.

Conclusion

Many techniques of fuzzy sets and systems theory were applied in various directions, such as: pattern recognition, decision-making under uncertainty, information retrieval, large-systems control, management science, and others. Some of these applications are described in greater detail in the book by Negoita and Ralescu, (1975). The theorem in this paper has its counterpart in classical measure theory which is also use in probability theory [Rudin (1974), Lee (2005)]. It should be noted that subadditivity condition cannot be dropped. The focus in this paper is therefore to obtaining some properties of fuzzy integral defined using the fuzzy measures in fuzzy sets such as additivity, positivity, transitive and other important properties..

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