EXISTENCE AND UNIQUENESS OF SOLUTION FOR CONTRACTION PRINCIPLE IN METRIC SPACES

Wahab, O. T.[1]; Rauf, K.[1]; Ezeibe, A. E.[2]; Omolehin, J. O.[1]; & Ma'ali, A. A. I.[3]
[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
[2]Department of Mathematics and Statistics, Federal University of Technology, Minna, Nigeria.
[3]Department of Mathematics & Computer Science, Ibrahim Babangida University Lapai, Nigeria
E-mail:
Phone No:

Abstract
This article uses contraction mapping principle in metric space to illustrate the existence and uniqueness of solution to second order differential equations. Some examples are considered to justify our claim.

Keywords: Picard theorem, Fixed point, Lipschitz map, Nonlinear operator.

Introduction

The study of nonlinear operator was introduced in the early twenties. The Picard's existence and uniqueness of solution to first-order equations [Picard (1893)] with given initial conditions have received rigorous attention of researchers. The proof is basically on transforming the differential equation and applying fixed point. It can be established by using the Banach fixed point theorem [A'lvarez (2011) and Banach (1932)] such that the Picard iteration is convergent with a unique limit. In this paper, we consider second order differential equations which is transformed to first-order vector differential equation and employ the Banach's theorem to discuss the existence and uniqueness of their solutions. See [Ambrosetti & A'lvarez (2011), Kreyszig, Maddox, Rhoades(1977)].

Preliminary Results

Let us consider the general first order equation:

$$y' = f(t,y) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

Where f is defined for (t,y) on some continuous sets. Suppose $f_1, \ f_2, \ \ldots, \ f_n$ are continuous-valued functions defined for $(t, y_1, \ y_2, \ \cdots \ , \ y_n)$ space. A wide class of (1) is the system:

$$\begin{aligned} y_1' &= f_1(t, y_1, \ y_2, \ \cdots \ , \ y_n) \\ y_2' &= f_2(t, y_1, \ y_2, \ \cdots \ , \ y_n) \\ \vdots \quad & \vdots \qquad\qquad\qquad \vdots \\ y_n' &= f_n(t, y_1, \ y_2, \ \cdots \ , \ y_n) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2) \end{aligned}$$

This is a system of n ordinary differential equations of the first order, the derivatives $y_1', y_2', y_3' \cdots\cdots\cdots\cdots, y_n'$ appear explicitly and they are analogue of (1).

Second Order Equation. An equation of second order

$$y'' = f(t, y, y') \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3)$$

may be treated as a system of the form (2).

Let y $= y_1$, $y' = y_2$

Then (3) can be written as:

$$y_1' = y_2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4a)$$
$$y_2' = f(t, y_1, y_2) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4b)$$

which may be viewed as the type (2). The clear difference between (1) and (2) is that a complex number y is now to deal with two such complex numbers y_1, y_2.

Let y be a vector of the two complex numbers and we may write

y = (y_1, y_2). The set of all such vectors is called the complex

2-dimensional space C^2. Systems as Vector Equations. Consider the first order system of equations

$$y_1' = f_1(t, y_1, y_2) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5a)$$
$$y_2' = f_2(t, y_1, y_2) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5b)$$

It is assumed that f_1, f_2 are complex-valued functions defined for

(t, y_1, y_2) on some set, where t is real and y_1, y_2 are complex.

Clearly, f_1, f_2 are functions of t and the vector y, where y = (y_1, y_2) in C^2. Therefore, we may write

$$f_1(t, y) = f_1(t, y_1, y_2)$$
$$f_2(t, y) = f_2(t, y_1, y_2)$$

In (5a) and (5b), we have two functions f_1, f_2 which may be regarded as a vector-valued function f = (f_1, f_2), which may also be given by

$$f(t, y) = (f_1(t, y), f_2(t, y)).$$

Suppose

$$y' = (y_1', y_2''),$$

then the system (5a) and (5b) may now be written as

$$y' = f(t, y) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (6)$$

Remark. The vector differential equation (6) now has the form (1).

Definition 2.1. A vector-valued function f is said to satisfy a Lipschitz condition on Ω if there is a number K > 0 such that

$$|f(t, y) - f(t, z)| \leq K\,|y - z| \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7)$$

for all y, z $\in$ C^2 and (t, y), (t, z) $\in$ Ω. The least value of constant K is called the Lipschitz constant.

Proposition 2.1. Let *f* be a vector-valued function defined for (t, y) on a set Ω given by

$$\Omega := \{(t, y) : |t - t_o| \leq a, |y - y_o| \leq b,\ a, b > 0\}$$

If $\frac{\partial f}{\partial yk}$ (k = 1, 2) is continuous on Ω and there is a constant $K > 0$ such that $\left|\frac{\partial f}{\partial yk}\right| \leq K$ for (t, y) $\in \Omega$, then f satisfies a Lipschitz condition on Ω.

Proof: See [Coddington(1989)]

Proposition 2.2. Consider the vector differential equation

$$y' = f(t,y)$$

where the components f_1, f_2 of f are of the form

$$f_1(t, y) = a_{11}(t)y_1 + a_{12}(t)y_2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (8a)$$
$$f_2(t, y) = a_{21}(t)y_1 + a_{22}(t)y_2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (8b)$$

where a_{11}(t),$\cdots$, a_{22}(t), b_1(t), b_2(t) are complex-valued functions defined for real t in some interval I. If all the aij are continuous on an interval I : $|t - t_o| \leq a$, where a > 0, then the corresponding vector-valued function f satisfies a Lipschitz condition on the strip

$$\Omega: \ |t - t_o| \leq a, |y - y_o| \leq b \ or \ |y| < \infty, \text{ a, b} > 0$$

Proof: See [Coddington(1989)]

Proposition 2.3. The vector differential equation (6) defined on Ω is equivalent to the integral equation

$$y = y_o + \int_{t_o}^{t} f(\tau, y(\tau))d\tau \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (9)$$

where $y_o = (\alpha_1, \alpha_2), f(\tau, y(\tau)) = (f_1, f_2)$ and

$$f_k(\tau, y(\tau)) = \sum_1^2 a_{jk}(\tau)\, y_k(\tau) + b_k(\tau), K = 1,2$$

We complete this section with a proposition which is sequel to our work.

Proposition 2.4. Let X be a metric space. Then X is said to be complete if every cauchy sequence in X has a limit x which is in X.
A subset Y of a metric space X is complete if it is closed See [Chidume (1989)].

Problem Formulation

In this section, we discuss the Banach fixed point theorem which states sufficient conditions for the existence and uniqueness of a fixed point and also gives a constructive procedure for obtaining sharp results to the fixed point. We start with the following definitions:

Definition 3.1. Let X be a non-empty set and T be a mapping of X into itself. A point $x \in X$ is said to be a fixed point of the mapping T if

$$Tx = x \text{ ....(10)}$$

i.e. the image Tx coincides with x

Definition 3.2. Let $X = (X, d)$ be a metric space. A mapping $T: X \to X$ is called a Lipschitz map if there is a real number $c > 0$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq cd(x, y) \qquad \text{.....(11)}$$

for all $x, y \in X$ and T is called a contraction on X if there is a positive real number $c < 1$ such that for all $x, y \in X$.

Remark. If $c = 1$, then (11) becomes $d(Tx, Ty) < d(x, y)$ which may not be replaced for (11). In this case, T is called nonexpansive (9).

Proposition 3.3. Let T be a contraction mapping, then for any positive integer n, T^n is also a contraction mapping.

Proof

Let T be a contraction mapping $T: X \to X$, (by Definition 3.2) there exists $c < 1$ for $x, y \in X$ such that

$$d(Tx, Ty) \leq cd(x, y)$$

Now,

$$\begin{aligned} d(T^n x, T^n y) &= d\big(T(T^{n-1}x), T(T^{n-1}y)\big) \\ &\leq cd(T^{n-1}x, T^{n-1}y) \\ &\leq c^2 d(T^{n-2}x, T^{n-2}y) \\ &\;\;\vdots \\ &= c^n d(T^{n-n}x, T^{n-n}y) \\ &\leq c^n d(x, y) \\ \Rightarrow d(T^n x, T^n y) &\leq c^n d(x, y) \end{aligned}$$

Since $c < 1$, then $c^n < 1$ for all n. Therefore, T^n is a contraction.

Remark. If c is a constant of contraction T then c^n is a constant of contraction T^n.

Proposition 3.4. Every contraction mapping of a metric space (X, d) is a continuous mapping.

Proof

Let $T: X \to X$ be a contraction mapping of a metric space X, then there is a positive constant $c < 1$ such that

$$d(Tx, Ty) \leq cd(x, y) \text{ for all } x, y \in X$$

Let $\varepsilon > 0$ be given, we want to find $\delta > 0$ such that whenever

$$d(x, y) < \delta \Rightarrow d(Tx, Ty) < \varepsilon$$

Choose $0 < \delta < \frac{\varepsilon}{c}$. Then, for $x, y \in X$ $d(x, y) < \delta$

$$\Rightarrow d(Tx, Ty) \leq cd(x, y) < c\frac{\varepsilon}{c} = \varepsilon$$

Hence the proof. See [1] for similar proof.

Theorem 3.5 (Banach Fixed Point Theorem). Let X be a non empty metric space. Suppose that X is complete and $T: X \to X$, is a contraction on X. Then, T has precisely one fixed point $x \in X$.

Proof

Let $x_0 \in X$ be arbitrarily fixed and define the iterative sequence $\{x_n\}$ by

$$x_0, x_1 = Tx_0,\ x_2 = T^2x_0, \ldots, x_n = T^nx_o. \text{ ....(12)}$$

We have constructed the sequence of various images of x_0 under repeated application of T. Next, we show that $\{x_n\}$ is Cauchy.

By (10) and (11), we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) & \\ & \leq cd(x_{n-1}, x_n) \\ & \leq c^2 d(x_{n-2}, x_{n-1}) \\ & \vdots \\ & = c^n d(x_{n-n}, x_{n-n+1}) \\ & \leq c^n d(x_0, x_1) \quad \text{..................... (13)} \end{aligned}$$

Let $m > n$ for $n, m \in N$, then by geometric progression and proposition (3.3),we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$

$$\leq c^n d(x_0, x_1) + c^{n+1} d(x_0, x_1) + \cdots + c^{m-1} d(x_0, x_1)$$
$$= c^n d(x_0, x_1)(1 + c + c^2 + \cdots + c^{m-n-1})$$
$$= c^n \left(\frac{1-c^{m-n}}{1-c}\right) d(x_0, x_1)$$

Since $c < 1$, then $1 - c^{m-n} < 1$ for $m - n > 0$

So that,

$$d(x_n, x_m) \leq \left(\frac{c^n}{1-c}\right) d(x_0, x_1) \quad \text{.................. (14)}$$

On the right, $c < 1$ and $d(x_0, x_1)$ is fixed. So, as $n \to \infty, c^n \to 0$ which make the right hand side inequality as small as we please.

This proves that $\{x_n\}$ is Cauchy.

Since X is a complete metric space, then $\{x_n\}$ converges to a point, say, x in X, that is

$$x_n \to x, as\ n \to \infty \quad \text{...................................... (15)}$$

Also, since T is a contraction, (by proposition (3.4)) T is continuous.

Therefore, $Tx_n \to Tx$ whenever (15) holds.

Next is to show that the limit x is the fixed point of the mapping T.

By (10),

$$d(Tx, x) \leq d(x, x_n) + d(x_n, Tx)$$
$$\leq d(x, x_n) + cd(x_{n-1}, x)$$

By (15), $x_n \to x$ and $x_{n-1} \to x$, as $n \to \infty$

Thus,

$$d(Tx, x) = 0 \Leftrightarrow Tx = x$$

And finally, we show that the limit x is the only fixed point of T.

Suppose x and $\tilde{x}$ are two fixed points, then

$$d(x, \tilde{x}) = d(Tx, T\tilde{x})$$
$$\leq cd(x, \tilde{x})$$

Thus,

$$d(x, \tilde{x}) = 0, \text{ if and only if } x = \tilde{x}$$

Hence, x is the only fixed point of T.

This completes the proof.

Corollary 3.6. Let X be a complete metric space and T is such that $T: X \to X$. Suppose T^n is a contraction on X, then T^n has only one fixed point.

Remark. Generally in application, the mapping T is a contraction not on the entire space X. Since a closed subset of a complete space X is complete, T has a fixed point on the closed

subset provided there is a restriction on the choice of x_0 so that the x_n lie in the closed subset.
This is justified by the following theorems.
Theorem 3.7 (Baire Category). Let X be a non empty complete metric space. Let $\{F_n\}_{n=1}^{\infty}$ be sequence of closed sets such that

$$X = \bigcup_{n=1}^{\infty} F_n$$

Then, there exists an integer n_0 such that $Int(F_{n_0}) \neq \emptyset$

Proof

Suppose no F_n contains an open ball. Let S_0 be an open ball in X , then for any $x_0 \in S_0$, there exists $r_0 < 1$ such that

$$B(x_0, r_0) \subset S_0$$

Then, the complement F_n^c intersects every open ball $S_0 \in X$.
Observe that $F_n^c \cap S_0$ is a non empty open set.
Let $x_1 \in F_1^c \cap S_0$, there exists $r_1 < \frac{1}{2}$ such that $\overline{B(x_1, r_1)} \subset F_1^c \cap S_0$
Since no F_n contains an open ball, then $B(x_1, r_1) \not\subset F_1$
Hence, F_2^c intersects B_1. Let $x_2 \in F_2^c \cap B_1$, since $F_2^c \cap B_1$ is an open set, then $\exists\, r_2 < \frac{1}{4}$ such that

$$B_2 \equiv \overline{B(x_2, r_2)} \subset F_2^c \cap B_1 \subset B_1$$

Since no F_n contains an open ball, then $B(x_2, r_2) \not\subset F_3$

$$F_3^c \cap B_2 \neq \emptyset$$

Let $x_3 \in F_3^c \cap B_2$, since $F_3^c \cap B_2$ is an open set, then $\exists\, r_3 < \frac{1}{8}$ such that

$$B_3 \equiv \overline{B(x_3, r_3)} \subset F_3^c \cap B_2 \subset B_2$$

Since no F_n contains an open ball, then $B(x_2, r_2) \not\subset F_3$, if we continue in this manner , we obtain , by induction a sequence $B_n \equiv B(x_n, r_n)$

$$B_{n+1} \equiv \overline{B(x_{n+1}, r_{n+1})} \subset F_{n+1}^c \cap B_n \subset B_n, \quad r_n < \frac{1}{2^n}$$

Furthermore, $\overline{B_{n+1}} \subset B_n$, for each n
Moreover, $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequencein X. By completeness of X.
$x_n \to x^* \in \bigcap_{n=1}^{\infty} B_n$ (i.e $x^* \in B_n$ $for\ each\ n$) Since F_n^c intersects B_n for each n.
Hence,

$x^* \in F_n^c \Longrightarrow x^* \notin F_n$ for each n

This implies

$$x^* \notin \bigcup_{n=1}^{\infty} F_n$$

This is a contradiction and therefore

$$Int(F_{n_0}) \neq \emptyset$$

Remark. Observe that the diameter of $B_1 \supset B_2 \supset B_3 \supset. \ . \ . \supset B_n \supset B_{n+1}$, where each B_n is a non empty closed subset of X, shrinks to a point , i.e diameter $(B_n) \to 0 \ \ as\, n \to \infty$ and we have

$\bigcap_{n=1}^{\infty} B_n \neq \emptyset$ ....(16)

In fact, (16) is a singleton set. This is generally reffered to as Principle of Nested sequences. We shall show how this theorem can be adapted to show existence and uniqueness of solutions of vector differential equation (6)

Main Result

We begin with the following theorem

Theorem 4.1. Let f be a continuous vector-valued function defined on

$$\Omega := \{(t, y) : |t - t_0| \leq a, |y - y_0| \leq b, (a, b > 0)\}$$

And bounded on Ω, say

$$|f(t, y)| \leq M$$

Suppose f satisfies a Lipschitz condition on Ω with respect to its second argument. Then, the iterative function sequence $\{\Phi_m\}_{m=1}^{\infty}$ obtained in (19) converge on the interval $[t_0 - \beta, t_0 + \beta]$ where

$\beta < min\left\{a, \frac{b}{M}, \frac{1}{K}\right\}$...(17)

To a solution Φ of the system (6)

Proof

Let $C(I)$ be the metric space of all complex-valued continuous function on the interval $I = [t_0 - a, t_0 + a]$. For $t \in [t_0 - a, t_0 + a]$ and $\Phi(t), \Psi(t) \in C(I)$, the metric on $C(I)$ is defined by

$$d(\Phi(t), \Psi(t)) = \sup_{t \in [t_0 - a, t_0 + a]} |\Phi(t) - \Psi(t)|$$

$C(I)$ is complete (7).

Let $J = [t_0 - \beta, t_0 + \beta] \subset I$, then $C(J)$ is a closed subspace of $C(I)$ which is also complete by proposition 2.5

Define the mapping $T: C(J) \to C(J)$ and $T\Phi(t) = \Phi(t)$ for $\Phi \in C(J)$

Consider a ball B in $C(J)$ with radius b centred at y_0 given by

$B = \{\Phi \in C(J): |\Phi(t) - y_0| \leq b\}$

We show that $B \supset T(B)$, suppose

$$T\Phi(t) = y_0 + \int_{t_0}^{t} f_k(\tau, \Phi(\tau))d\tau$$

Where

$$f_k(\tau, y(\tau)) = \sum_{j=1}^{2} a_{jk}(\tau)yk(\tau) + bk(\tau), \quad k = 1,2$$

$$\Longrightarrow d(T\Phi(t), y_0) = \sup |T\Phi(t) - y_0|$$

$$= sup\left|\int_{t_0}^{t} f_k(\tau, \Phi(\tau))d\tau\right|$$

$$\leq sup\left|\int_{t_0}^{t} |f_k(\tau, \Phi(\tau))|d\tau\right|$$

$$\leq M\ sup|t - t_0|$$

$$\leq M\beta < b.$$

Which implies for $\Phi \in T(B) \Longrightarrow \Phi \in B$, and thus, T maps $C(J)$ into itself. Next is to show that T is a contradiction on $C(J)$.

By the Lipschitzian assumptions (7) and for $\Phi(\mathrm{t}), \Psi(t) \in C(J)$.

We have

$$d(T\Phi, T\Psi) = sup|T\Phi(\mathrm{t}) - T\Psi(\mathrm{t})|$$

$$= sup\left|\int_{t_0}^{t} f_k(\tau, \Phi(\tau))d\tau - \left(\int_{t_0}^{t} f_k(\tau, \Psi(\tau))d\tau\right)\right|$$

$$\leq sup\left|\int_{t_0}^{t}\left|\sum_{j=1}^{2} a_{jk}(\tau)\phi_k(\tau) - \sum_{j=1}^{2} a_{jk}(\tau)\psi_k(\tau)\right| d\tau\right|$$

$$\leq sup\left|\int_{t_0}^{t}\sum_{j=1}^{2}|a_{jk}(\tau)||C\phi_k(\tau) - \psi_k(\tau)|d\tau\right|$$

$$\leq K\ sup|\Phi(\tau) - \Psi(\tau)|\ \mathrm{sup}|t - t_0|$$

$$\leq K\beta d(\phi, \psi)$$

From (17), choose $c = k\beta < 1$, so that T is a contraction on $c(J)$

The conclusion of the theorem follows from Theorem 3.5.

Remark 4.1

Observe that the existence result proved above is local. Moreso, interval I depends on M, K and on the initial condition.

Remark 4.2

Let f be a continuous vector –valued function and global on the strip.

$$\Omega' := \{(t, y) : |t - t_0| \leq a, |y| < \infty\}$$

Then the iterative sequence $\{\Phi_m(t)\}_{m=1}^{\infty}$ exist on $|t - t_0| \leq a$ and converge to a solution of the system (6).

We now discuss the existence and uniqueness of solution of a second order differential equation. We now consider the following examples:

Example 1:

Let us consider the problem

$$\frac{d^2u}{dt^2} + \mu^2 u = 0, \mu \in \Re, u(0) = 0, u'(0) = 1$$

Solution

Let u $= u_1$, so that

$$u_1' = u_2 \, ,$$
$$u_2' = -\mu^2 u_1$$

This can be represented as the vector differential equation

$$u' = f(t, u_{1,} u_2)$$

Where $f(t, u_1, u_2)$ $=(u_{2,-}\mu^2 u_1)$.

Now,

$$\frac{\partial f_1}{\partial u_1} = 0, \ \frac{\partial f_1}{\partial u_2} = 1$$
$$\frac{\partial f_2}{\partial u_1} = -\mu^2, \frac{\partial f_2}{\partial u_2} = 0$$

So,

$$\left|\frac{\partial \boldsymbol{f}}{\partial u_1}\right| = \mu^2, \left|\frac{\partial \boldsymbol{f}}{\partial u_1}\right| = 1$$

Thus, $\boldsymbol{f}$ satisfies a lipschitz conditions with constant $L = \mu^2$

Also, Let T be a mapping defined by

$$T\mathbf{u} = \mathbf{u_0} + \int_{t_0}^{t} \mathbf{f}\big(\tau, \mathbf{u}(\tau)\big) d\tau$$
$$d(T\mathbf{u}, T\mathbf{v}) = |T\mathbf{u}(t) - T\mathbf{v}(t)|$$

$$= \left| \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{u}(\tau)) d\tau - \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{v}(\tau))\, d\tau \right|$$

$$= \left| \int_{t_0}^{t} [\mathbf{f}(\tau, \mathbf{u}(\tau)) - \mathbf{f}(\tau, \mathbf{v}(\tau))] d\tau \right|$$

$$\leq \left| \int_{t_0}^{t} \left|(u_2, \mu^2 u_1) - (v_{2,}\mu^2 v_1)\right| d\tau \right|$$

$$\leq \left| \int_{t_0}^{t} |u_2 - v_2, \mu^2(v_1 - u_1)|\, d\tau \right|$$

$$= |t|(|u_2 - v_2| + |\mu^2||v_1 - u_1|)$$

$$\leq |t|(1 + \mu^2)|\mathbf{u} - \mathbf{v}|$$

$$= |t|K|\mathbf{u} - \mathbf{v}|$$

Where $K = 1 + \mu^2 \equiv 1 + L$ and $c = |t|K < 1$

Hence, T is a contraction

Now, we show that $\mathbf{u}_m \to \mathbf{u}, \quad m = 1,2,3, \ldots$

Let $\mathbf{u}_m \equiv \mathbf{u}^m and\ \mathbf{u}^0 = (0,1)$ be fixed.

$$\mathbf{u}^1 = (0,1) + \int_0^t \mathbf{f}(s, u_1^0 u_2^0) ds$$

$$= (0,1) + \int_0^t (u_2^0, -\mu u_1^0)\, ds$$

$$= (0,1) + \int_0^t (0,1) ds$$

$$= (0,1) + (t, 0) = (t, 1)$$

$$\mathbf{u}^2 = (0,1) + \int_0^t (\, u_2^1, -\mu u_1^1) ds$$

$$= (0,1) + \int_0^t (1, -\mu^2 s) ds$$

$$= (0,1) + \left(t, -\mu^2 \frac{t^2}{2}\right) = \left(t, 1 - \mu^2 \frac{t^2}{2}\right)$$

$$\mathbf{u}^3 = (0,1) + \int_0^t (\, u_2^2, -\mu u_1^2) ds$$

$$= (0,1) + \int_0^t \left(1 - \mu^2 \frac{s^2}{2}, -\mu^2 s\right) ds$$

$$= (0,1) + \left(t - \mu^2 \frac{t^3}{6}, -\mu^2 \frac{t^2}{2}\right) = \left(t - \mu^2 \frac{t^3}{6}, 1 - \mu^2 \frac{t^2}{2}\right)$$

$$\boldsymbol{u}^4 = (0,1) + \int_0^t (u_2^3 - \mu u_1^3) ds$$

$$= (0,1) + \int_0^t \left(1 - \mu^2 \frac{s^2}{2}, -\mu^2 s + \mu^4 \frac{s^3}{6}\right) ds$$

$$= (0,1) + \left(t - \mu^2 \frac{t^3}{6}, -\mu^2 \frac{t^2}{2} + \mu^4 \frac{t^4}{24}\right)$$

$$= \left(t - \mu^2 \frac{t^3}{6}, 1 - \mu^2 \frac{t^2}{2} + \mu^4 \frac{t^4}{24}\right)$$

$$\boldsymbol{u}^5 = (0,1) + \int_0^t (u_2^4, -\mu u_1^4) ds$$

$$= (0,1) + \int_0^t \left(1 - \mu^2 \frac{s^2}{2} + \mu^4 \frac{t^4}{24}, -\mu^2 s - \mu^4 \frac{s^3}{6}\right) ds$$

$$= (0,1) + \left(t - \mu^2 \frac{t^3}{6} + \mu^4 \frac{t^5}{120}, -\mu^2 \frac{t^2}{2} + \mu^4 \frac{t^4}{24}\right)$$

$$= \left(t - \mu^2 \frac{t^3}{6} \mu^4 \frac{t^5}{120}, 1 - \mu^2 \frac{t^2}{2} + \mu^4 \frac{t^4}{24}\right)$$

It is easily seen that $\mathbf{u}_m$ exists for all real t and that

$$\mathbf{u}_m \rightarrow \mathbf{u} = (sin\, \mu t, cos\, \mu t)$$

Example 2
Let us consider the problem

$$\frac{du}{dt} + u^{\frac{3}{4}} = 0, \qquad u(0) = 0$$

Solution

$$\Rightarrow \frac{du}{dt} = -u^{\frac{3}{4}} \equiv f(t,u)$$

Now,

$$\frac{\partial f}{\partial u} = -\frac{3}{4}u^{-\frac{1}{4}}$$

And

$$\left|\frac{\partial f}{\partial u}\right| = \frac{3}{4}u^{-\frac{1}{4}}$$

Observe that $f(t,u) = -u^{\frac{3}{4}}$ is not Lipchitian at the origin and hence, the uniqueness of the solution is not guaranteed.

Conclusion

In conclusion, if we suppose f is a continuous vector-valued function defined on

$$\widehat{\Omega} := \{(t,y): |t| < \infty, |y| < \infty\}$$

and satisfies Lipschitz conditions on each strip

$$(t,y): |t| \leq a, |y| < \infty$$

where a a is any positive number. Then, the iterative sequence

$$\{\Phi_m(t)\}_{m=1}^{\infty}$$

converges to a solution which exist for all real t.

References

Ambrosetti, A. & A'lvarez, A. (2011). An introduction to nonlinear functional analysis and elliptic promplems, progress in nonlinear differential equations and their applications 82, *Springer Science + Business Media*, LLC, XII, p199. 12 illus.

Banach, S. (1932). The'orie des operations lineires, *Warsaw publishing house*.

Caccioppoli, R. (1931). Un teorem generale sull'esistenza di elementi uniti in una transformazione, *Rend. Accad. Naz. Lincei*, 30, 498-502.

Chidume, C. E. (1989). An Introduction to metric spaces. Ibadan: *Longman, Nigeria Ltd.*

Coddington, E. A. (1989). An introduction to ordinary di_erential equations. *Dover Publisher*.

Dutta, P. N. & Choudhury, B. S. (2008). A generalization of contraction principle in metric spaces. *Fixed point theory and Appl.* Article ID406368, 1-8.

Kreyszig, E. (1978). Introductory Functional Analysis with Applications. New York: *John Wiley and Sons*.

Maddox, I. J. (1988). Elements of Functional analysis. USA: *Cambrigde University Press*.

Picard, E. (1893). Sur l'application des methodes d'approximations successive a l'etude de certaines equations differentielles ordinaires. *J. Math.*, 9, 217-271.

Rhoades, B. E. (1977). A comparison of various definitions of contractive mappings. *Transaction of the American Mathematical Society*, 226, 257-290.