MULTIVARIATE EXTENSION TO SUDOKU SQUARE DESIGN'S MODELS

SHEHU. A[1]. & DANBABA, A[2].
[1]Department of Mathematical Sciences& I.T Federal University,Dutsinma
[2]Department of Mathematics, Usmanu Danfodiyo University, Sokoto
E-mail: ashehu@fudutsinma.edu.ng **Phone No:** +234-803-629-8236

Abstract
Sudoku square design consists of treatments that are arranged in a square array such that each row, column or sub-square of the design contains each of the treatments only once. Several univariate analysis of variance models were presented in the literature for the design, but little or no attention is paid on the multivariate or generalized linear model for the design. This paper proposed multivariate extension of the analysis of variance (MANOVA) for the design and its generalized linear model. The significant tests were carried out at 0.05 alpha level of significance and the results show that effects are significant for Hui-Dong and Ru-Gen model and model II.

Introduction
A Sudoku square design is an experimental design with k^2 experimental units that are divided into k rows, k columns, and k boxes (i.e, each box contains k experimental units with 1 through k treatments). In this design each treatment has k replications (Hui-Dong and Ru-Gen, 2008). Detail description on how to design and randomize a Sudoku Square are presented in Hui-Dong and Ru-Gen (2008). However, the Sudoku design presented by Hui-Dong and Ru-Gen (2008) does not contain row-blocks or column-blocks effects. Subramani and Ponnuswamy (2009) extended the design to include row-blocks and column-blocks effects in which they called Sudoku designs-Type I. Methods of constructing the Sudoku design and analysis are discussed by Hui-Dong and Ru-Gen (2008), Subramani and Ponnuswamy (2009), and Danbaba (2016).

The construction of orthogonal Sudoku design has been considered by statisticians and mathematicians. Bailey *et al*. (2008) presented several results for construction of orthogonal Sudoku designs. Subramani (2012) extended the Sudoku designs to Orthogonal (Graeco) Sudoku square designs. A simple method of constructing Graeco Sudoku square designs of odd order is presented by Subramani (2012). Danbaba (2016a) presented a simple row (or column) permutation of matrix for construction of Graeco Sudoku square designs that does not require coding of treatments. Recently, it has been shown that Sudoku design may be partial or incomplete, see Béjar *et al*. (2012) and Mahdian and Mahmoodian (2015).

A partial Sudoku design is a partially filled block matrix, with some empty cells, which also satisfies that each Latin letter appears only once in row (column or sub-block), see Mahdian and Mahmoodian (2015). This partial Sudoku design has been shown to be NP-complete for the particular case of square sub-blocks (k rows and k columns in each sub-block), see Kanaana and Ravikumar (2010). It was reported in Béjar *et al*. (2012) that even when sub-blocks are not square the completion problem is also NP-complete. In general, it is an NP-complete problem to determine if a partial Sudoku square is completable (Colbourn 1984; Mahdian and Mahmoodian 2015).

Donovan *et al*. (2015) and Kumar *et al*. (2015) studied the Sudoku based space filling designs. Recently, Danbaba (2016b) proposed combined analysis of multi-environment experiments conducted via Sudoku square designs of odd order where the treatments are the same to the whole set of experiments. Danbaba (2016a) proposed a simple method of constructing Samurai Sudoku designs and orthogonal (Graeco) Samurai Sudoku design. He also discussed linear models and methods of data analysis for these designs. The linear models and methods of data analysis for all the Sudoku designs considered so far are on univariate analysis of variance.

This paper proposed a multivariate procedure for the analysis of variance of Sudoku design. Sum of squares for analysis of variance of the design are presented and illustrated with some hypothetical data.

Method

Hui-Dong and Ru-Gen (2008) proposed the following linear model for Sudoku square design as

$$y_{(ij)lm} = \mu + \alpha_i + \beta_j + \gamma_l + \theta_m + \varepsilon_{(ij)lm} \begin{cases} i = 1, 2, \ldots, k \\ j = 1, 2, \ldots, k \\ l = 1, 2, \ldots, k \\ m = 1, 2, \ldots, k \end{cases} \quad (1)$$

where $y_{(ij)lm}$ is an observed value of the plot in the *l*th row and *m*th column, subjected to the *i*th treatment and *j*th box; μ is the grand mean, $\alpha_i, \beta_j, \gamma_l, \theta_m$ are the main effects of the *i*th treatment, *j*th box, *l*th row and *m*th column, respectively, ε_{ijm} is the random error.

The proposed multivariate extension of that model is

$$\mathbf{y}_{(ij)lm} = \boldsymbol{\mu} + \boldsymbol{\alpha}_i + \boldsymbol{\beta}_j + \boldsymbol{\gamma}_l + \boldsymbol{\theta}_m + \boldsymbol{\varepsilon}_{(ij)lm} \quad (2)$$

where $\mathbf{y}_{(ij)lm}$ is a p-vector-valued observations and $\varepsilon_{(ij)m} \sim N_p(0, \Sigma)$

The MANOVA Table and hypotheses of interest are as follows:

Table 1: MANOVA table for Hui-Dong and Ru-Gen Model

Source	df	SSP
Treatments (α)	k-1	$D_\alpha = \sum_{i=1}^{k} \frac{y_{i\ldots}y'_{i\ldots}}{k} - \frac{y_{\ldots}y'_{\ldots}}{k^2}$
Boxes (β)	k-1	$D_\beta = \sum_{j=1}^{k} \frac{y_{j\ldots}y'_{j\ldots}}{k} - \frac{y_{\ldots}y'_{\ldots}}{k^2}$
Rows (γ)	k-1	$D_\gamma = \sum_{l=1}^{k} \frac{y_{l\ldots}y'_{l\ldots}}{k} - \frac{y_{\ldots}y'_{\ldots}}{k^2}$
Columns (θ)	k-1	$D_\theta = \sum_{l=1}^{k} \frac{y_{\ldots m}y'_{\ldots m}}{k} - \frac{y_{\ldots}y'_{\ldots}}{k^2}$
Error	(k-1)(k-1)	$D_\varepsilon = \sum_{l=1}^{k} y_{ijlm}y'_{ijlm} - \frac{y_i \; y'_i}{k^2}$
Total	k^2 -1	$D_T = \sum_{l=1}^{k} y_{ij(kl)}y'_{ij(kl)} - \frac{y_{\ldots}y'_{\ldots}}{k^2}$

H_α : all $\boldsymbol{\alpha}_i$ are equal

H_β : all $\boldsymbol{\beta}_i$ are equal

H_γ : all $\boldsymbol{\gamma}_i$ are equal

H_θ : all $\boldsymbol{\theta}_i$ are equal

Subramani and Ponnuswamy (2009) constructed and analyzed Sudoku designs of which four types of univariate models were suggested. In this paper the following multivariate alternative models are suggested for analysis of Sudoku designs of odd order:

Type 1

$$y_{ij(k,l,p,q)} = \mu + \alpha_i + \beta_j + \tau_k + C_p + \gamma_l + s_q + \varepsilon_{i,j(k,l,p,q)} \tag{3}$$

i, j = 1, 2, ...,*m and k, l, p, q* = 1, 2, ..., m^2
where μ= General mean
α_i = i^{th} Row block effect
β_j = j^{th} Column block effect
τ_k = k^{th} Treatment effect
r_l = l^{th} Row effect
c_p = p^{th} Column effect
s_q = q^{th} Square effect
$\varepsilon_{i,j(k,l,p,q)}$ = is the error component assumed to have vector mean zero and constant covariance Σ

Let $Y_{..j..}$ be the row-box (or row block) total and $Y_{i.j..}$ be the column-box (or column block) totals. The respective sum of squares and product matrices for row-block and column-block are

$$D_{\alpha\theta} = \sum_{i=1}^{k} \frac{y_{i...}y'_{i...}}{m^2} - \frac{y_{...}y'_{...}}{m^3}$$

$$D_{\beta\theta} = \sum_{j=1}^{k} \frac{y_{j...}y'_{j...}}{m^2} - \frac{y_{...}y'_{...}}{m^3} \tag{4}$$

The respective sum of squares and product matrices for rows within row-block and column within column-block are

$$D_{\alpha(\alpha\theta)} = \sum_{i=1}^{k} \frac{y_{l(i)...}y'_{i...}}{m^2} - \sum_i \frac{y_{...i}y'_{...i}}{m^3} \tag{5}$$

$$D_{\beta(\beta\theta)} = \sum_{j=1}^{k} \frac{y_{p(j)...}y'_{j...}}{m^2} \quad \sum_{i} \frac{y_{..j}y'_{..j}}{m^3}$$

The respective sum of squares and product matrices for boxes within row-block and boxes within column-block are

$$D_{\beta(\beta\theta)} = \sum_{i=1}^{k} \frac{y_{r(j)...}y'_{r(j)...}}{m^2} - \sum_{m} \frac{y_{...j}y'_{..j}}{m^3}$$

$$D_{\alpha(\alpha\theta)} = \sum_{j=1}^{k} \frac{y_{q(i)...}y'_{q(i)...}}{m^2} - \sum_{m} \frac{y_{..i}y'_{..i}}{m^3} \tag{6}$$

Table 2: MANOVA Table of Sudoku design of Type I

Source	df	SSP
Treatments	k – 1	D_γ
Row blocks	1	$D_{\alpha\theta}$
Column block	1	$D_{\beta\theta}$
Rows	k – 1	D_α
columns	k – 1	D_β
Boxes	k – 1	D_θ
Error	1	
Total	k^2 – 1	D_T

Type II

$$y_{ij(k,l,p,q)} = \mu + \alpha_i + \beta_j + \tau_k + \gamma(\alpha)_{l(i)} + C(\beta)_{p(j)} + s_q + e_{i,j(k,l,p,q)} \quad (7)$$

$i, j, l, p = 1, 2, ...,m$ *and* $k, q = 1, 2, ...,m^2$

where μ= General mean
α_i = i^{th} block (Row) effect
β_j = j^{th} block (Column) effect
τ_k = k^{th} treatment effect
$r(\alpha)_{l(i)}$ = l^{th} row effect nested in i^{th} block (row)
$c(\beta)_{p(j)}$ = p^{th} column effect nested in j^{th} block (column)
s_q = q^{th} square effect
$\varepsilon_{i,j(k,l,p,q)}$= the error component assumed to have vector mean zero and constant covariance Σ

Table 3: MANOVA Table of Sudoku design of Type II

Source	df	SSP
Treatments	k – 1	D_γ
Row blocks	1	$D_{\alpha\theta}$
Column block	1	$D_{\beta\theta}$
Rows within row block	2	$D_{\alpha(\alpha\theta)}$
columns within column block	2	$D_{\beta(\beta\theta)}$
Boxes	k – 1	D_θ
Error	2	
Total	$k^2 - 1$	D_T

Type IV

$$y_{ij(k,l,p,q,r)} = \mu + \alpha_i + \beta_j + \tau_k + \gamma(\alpha)_{l(i)} + c(\beta)_{p(j)} + s(\alpha)_{q(i)} + \pi(\beta)_{r(j)} + \varepsilon_{ij(k,l,p,q,r)} \quad (8)$$

$i, j, l, p, q, r = 1, 2,. . .,m$ *and* $k = 1, 2,. . .,m^2$
where
μ = General mean
α_i = i^{th} Row block effect
β_j = j^{th}Column block effect
τ_k = k^{th} Treatment effect
$r(\alpha)_{l(i)}$ = l^{th} Row effect nested in i^{th} row block effect
$c(\beta)_{p(j)}$ = p^{th} Column effect nested in j^{th} column block effect
$s(\alpha)_{q(i)} - qth$ Horizontal square effect nested in ith Row block effect
$\pi(\beta)_{r(j)} = rth$ vertical square effect nested in the jth column block effect
$\varepsilon_{ij(k,l,p,q,r)}$ = the error component assumed to have vector mean zero and constant covariance Σ

Table 4: MANOVA table of Sudoku design of Type IV

Source	df	SSP
Treatments	k – 1	D_{γ}
Row blocks	1	$D_{\alpha\theta}$
Column block	1	$D_{\beta\theta}$
Rows within row block	2	$D_{\alpha(\alpha\theta)}$
columns within column block	2	$D_{\beta(\beta\theta)}$
Boxes within row block	2	$D_{\theta(\alpha\theta)}$
Boxes within column block	2	$D_{\beta(\beta\theta)}$
Error	2	
Total	$k^2 - 1$	D_T

Muiltivariate Test Statistics used

All the null hypotheses ($H_{\alpha}, H_{\beta}, H_{\gamma}, H_{\theta}$) of main-effect are rejected at α level of significance if

(i) Roy Largest Root : $\frac{\lambda_a}{\lambda_a + 1} > \theta(\min(k-1, p), q, v),$

(ii) Lawley-Hoteling : $\sum_{i=1} \lambda_i > \theta(\min(k-1, p), q, v)$

(iii) Wilk's Lambda : $\Lambda = \frac{|D|}{|D + D|} < \Lambda(p, k-1, g)$

(iv) Pillai: $\sum_{i=1} \frac{\lambda_i}{\lambda_i + 1} > \theta(\min(k-1, p), q, v)$ (Timm,1975)

Where

g is the degree of freedom for error

$q = \frac{1}{2}(|k - 1 - p| - 1) \quad v = \frac{1}{2}(g - p - 1)$

Results

$y_{(ij)lm}$ is observation i^{th} treatment in the j^{th} box on the l^{th} row in mth column. Fig.1 shows the arrangement of treatments used for the study, it is a Sudoku Square design of order four. While Fig.2 revealed the dependent variable $y_{(ij)lm}$ each of these variables represent more than one responses in each plot i.e (Multivariate case). However, each of the $y_{(ij)lm}$'s in Fig 2 is written in a linear model, which together forms the generalized linear model as seen in equation 10.

		1: 1	1: 2	2: 1	2: 2
1	1	A	B	C	D
	2	D	C	B	A
2	1	B	A	D	C
	2	C	D	A	B

Fig.1: Sudoku Square design of order 4 showing treatment arrangement

		1		2	
		1	2	1	2
1	1	y_{1111}	y_{1221}	y_{1332}	y_{1442}
	2	y_{2141}	y_{2231}	y_{2322}	y_{2412}
2	1	y_{3123}	y_{3213}	y_{3344}	y_{3434}
	2	y_{4133}	y_{4243}	y_{4314}	y_{4424}

Fig. 2: Sudoku square design of order 4 showing dependent variable

$$\begin{bmatrix} y_{1111} \\ y_{1221} \\ y_{1332} \\ y_{1442} \\ y_{2141} \\ y_{2231} \\ y_{2322} \\ y_{2412} \\ y_{3123} \\ y_{3213} \\ y_{3344} \\ y_{3434} \\ y_{4133} \\ y_{4243} \\ y_{4314} \\ y_{4424} \end{bmatrix} = \begin{bmatrix} 1&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\ 1&1&0&0&0&0&1&0&0&0&1&0&0&1&0&0&0 \\ 1&1&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0 \\ 1&1&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0 \\ 1&0&1&0&0&1&0&0&0&0&0&0&1&1&0&0&0 \\ 1&0&1&0&0&0&1&0&0&0&0&1&0&1&0&0&0 \\ 1&0&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0 \\ 1&0&1&0&0&0&0&0&1&1&0&0&0&0&1&0&0 \\ 1&0&0&1&0&1&0&0&0&0&1&0&0&0&0&1&0 \\ 1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0 \\ 1&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&1 \\ 1&0&0&1&0&0&0&0&1&0&0&1&0&0&0&0&1 \\ 1&0&0&0&1&1&0&0&0&0&0&1&0&0&0&1&0 \\ 1&0&0&0&1&0&1&0&0&0&0&0&1&0&0&1&0 \\ 1&0&0&0&1&0&0&1&0&1&0&0&0&0&0&0&1 \\ 1&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} + \begin{bmatrix} \varepsilon_{1111} \\ \varepsilon_{1221} \\ \varepsilon_{1332} \\ \varepsilon_{1442} \\ \varepsilon_{2141} \\ \varepsilon_{2231} \\ \varepsilon_{2322} \\ \varepsilon_{2412} \\ \varepsilon_{3123} \\ \varepsilon_{3213} \\ \varepsilon_{3344} \\ \varepsilon_{3434} \\ \varepsilon_{4133} \\ \varepsilon_{4243} \\ \varepsilon_{4314} \\ \varepsilon_{4424} \end{bmatrix}$$

$$\mathbf{Y}_{16\times1} = \mathbf{X}_{16\times17}\ \boldsymbol{\beta}_{17\times1} + \boldsymbol{\varepsilon}_{16\times1} \quad (9)$$

The multivariate extension of this model is proposed as follows:

$$\mathbf{Y}_{16\times n} = \mathbf{X}_{16\times17}\ \boldsymbol{\beta}_{17\times n} + \boldsymbol{\varepsilon}_{16\times n} \quad (10)$$

Illustration

					$Y_{i..l}$	$Y_{i..}$
	A$(2\ \ 1)$	B$(1\ \ 3)$	C$(3\ \ 1)$	D$(2\ \ 5)$	$(8\ \ 10)$	$(20\ \ 20)$
	D$(1\ \ 3)$	C$(2\ \ 5)$	B$(5\ \ 1)$	A$(4\ \ 1)$	$(12\ \ 10)$	
	B$(2\ \ 3)$	A$(2\ \ 4)$	D$(3\ \ 4)$	C$(2\ \ 3)$	$(9\ \ 14)$	$(22\ \ 25)$
	C$(3\ \ 4)$	D$(4\ \ 3)$	A$(5\ \ 2)$	B$(1\ \ 2)$	$(13\ \ 11)$	
$Y_{.jk.}$	$(8\ \ 11)$	$(9\ \ 15)$	$(16\ \ 8)$	$(9\ \ 11)$	$(42\ \ 45)$	
$Y_{.j..}$	$(17\ \ 26)$		$(25\ \ 19)$			

Fig. 3: Hypothetical data for Sudoku square design of order 4.

Fig 3 is a hypothetical data used for the analysis, procedures for the estimation of various sums of squares and products (SSP) are obtained using the statistics in Table 1 as well as equations 4-6 and the summaries are presented in Tables 5 – 8.

The process of obtaining the multivariate significant tests for various effects for the models is that, since the tests described in section 2.1 are functions of eigenvalues λ_i, the values of these eigenvalues are obtained and the calculated values for the multivariate tests are as well obtained using statistics described in section 2.1 and the summaries of the significant tests for the models are also presented in Tables 9-24.

From the data above, we have

Treatment (trt) Total

A	B	C	D
(13 8)	(9 9)	(10 13)	(10 15)

Box Total

B1	B2	B3	B4
(6 12)	(14 8)	(11 14)	(11 11)

$cf = \frac{y_{...}y'_{...}}{k^2}$, $y_{...} = \begin{pmatrix} 42 \\ 45 \end{pmatrix}$

$$cf = \frac{\begin{pmatrix} 42 \\ 45 \end{pmatrix}\begin{pmatrix} 42 & 45 \end{pmatrix}}{4 \times 4} = \begin{pmatrix} 110.2500 & 118.1250 \\ 118.1250 & 126.5625 \end{pmatrix}$$

$$\sum y_{ijlm}^2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \end{pmatrix} + \ldots + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 136 & 108 \\ 108 & 155 \end{pmatrix}$$

$$SSP_{Total} = \begin{pmatrix} 136 & 108 \\ 108 & 155 \end{pmatrix} - cf = \begin{pmatrix} 25.750 & -10.1250 \\ -10.1250 & 28.4375 \end{pmatrix}$$

$$SStrt = \frac{\begin{pmatrix} 13 \\ 8 \end{pmatrix}\begin{pmatrix} 13 & 8 \end{pmatrix} + \ldots + \begin{pmatrix} 10 \\ 15 \end{pmatrix}\begin{pmatrix} 10 & 15 \end{pmatrix}}{4} - cf = \begin{pmatrix} 2.250 & -1.8750 \\ -1.8750 & 8.1875 \end{pmatrix}$$

$$SS_{row-block} = \frac{\begin{pmatrix} 20 \\ 20 \end{pmatrix}\begin{pmatrix} 20 & 20 \end{pmatrix} + \begin{pmatrix} 22 \\ 25 \end{pmatrix}\begin{pmatrix} 22 & 25 \end{pmatrix}}{8} - cf = \begin{pmatrix} 0.250 & 0.6250 \\ 0.6250 & 1.5625 \end{pmatrix}$$

$$SS_{column-block} = \frac{\begin{pmatrix} 17 \\ 26 \end{pmatrix}\begin{pmatrix} 17 & 26 \end{pmatrix} + \begin{pmatrix} 25 \\ 19 \end{pmatrix}\begin{pmatrix} 25 & 19 \end{pmatrix}}{8} - cf = \begin{pmatrix} 4.000 & -3.500 \\ -3.500 & 3.0625 \end{pmatrix}$$

$$SS_{row} = \frac{\begin{pmatrix} 8 \\ 10 \end{pmatrix}\begin{pmatrix} 8 & 10 \end{pmatrix} + \ldots + \begin{pmatrix} 13 \\ 11 \end{pmatrix}\begin{pmatrix} 13 & 11 \end{pmatrix}}{4} - cf = \begin{pmatrix} 4.250 & -0.8750 \\ -0.8750 & 2.6875 \end{pmatrix}$$

$$SS_{column} = \frac{\begin{pmatrix} 8 \\ 11 \end{pmatrix}\begin{pmatrix} 8 & 11 \end{pmatrix} + \ldots + \begin{pmatrix} 9 \\ 11 \end{pmatrix}\begin{pmatrix} 9 & 11 \end{pmatrix}}{4} - cf = \begin{pmatrix} 10.250 & -5.6250 \\ -5.6250 & 6.1875 \end{pmatrix}$$

$$SS_{boxes} = \frac{\begin{pmatrix} 6 \\ 12 \end{pmatrix}\begin{pmatrix} 6 & 12 \end{pmatrix} + \ldots + \begin{pmatrix} 11 \\ 11 \end{pmatrix}\begin{pmatrix} 11 & 11 \end{pmatrix}}{4} - cf = \begin{pmatrix} 8.250 & -3.3750 \\ -3.3750 & 4.6875 \end{pmatrix}$$

$$SS_{rowwithrowblock} = \frac{\begin{pmatrix} 8 \\ 10 \end{pmatrix}\begin{pmatrix} 8 & 10 \end{pmatrix} + \ldots + \begin{pmatrix} 13 \\ 11 \end{pmatrix}\begin{pmatrix} 13 & 11 \end{pmatrix}}{4} - \frac{\begin{pmatrix} 20 \\ 20 \end{pmatrix}\begin{pmatrix} 20 & 20 \end{pmatrix} + \begin{pmatrix} 22 \\ 25 \end{pmatrix}\begin{pmatrix} 22 & 25 \end{pmatrix}}{8}$$

$$= \begin{pmatrix} 4.000 & -1.5000 \\ -1.5000 & 1.125 \end{pmatrix}$$

$$SS_{columnwithincolumnblock} = \frac{\begin{pmatrix} 8 \\ 11 \end{pmatrix}\begin{pmatrix} 8 & 11 \end{pmatrix} + \ldots + \begin{pmatrix} 9 \\ 11 \end{pmatrix}\begin{pmatrix} 9 & 11 \end{pmatrix}}{4} - \frac{\begin{pmatrix} 17 \\ 26 \end{pmatrix}\begin{pmatrix} 17 & 26 \end{pmatrix} + \begin{pmatrix} 25 \\ 19 \end{pmatrix}\begin{pmatrix} 25 & 19 \end{pmatrix}}{8}$$

$$= \begin{pmatrix} 6.2500 & -2.125 \\ -2.125 & 3.125 \end{pmatrix}$$

$$SS_{H.boxeswithinrowblock} = \frac{\begin{pmatrix} 6 \\ 12 \end{pmatrix}\begin{pmatrix} 6 & 12 \end{pmatrix} + \ldots + \begin{pmatrix} 11 \\ 11 \end{pmatrix}\begin{pmatrix} 11 & 11 \end{pmatrix}}{4} - \frac{\begin{pmatrix} 20 \\ 20 \end{pmatrix}\begin{pmatrix} 20 & 20 \end{pmatrix} + \begin{pmatrix} 22 \\ 25 \end{pmatrix}\begin{pmatrix} 22 & 25 \end{pmatrix}}{8}$$

$$= \begin{pmatrix} 8.0000 & -4.0000 \\ -4.0000 & 3.125 \end{pmatrix}$$

$$SS_{V.boxeswithinrowblock} = \frac{\begin{pmatrix} 6 \\ 12 \end{pmatrix}\begin{pmatrix} 6 & 12 \end{pmatrix} + \ldots + \begin{pmatrix} 11 \\ 11 \end{pmatrix}\begin{pmatrix} 11 & 11 \end{pmatrix}}{4} - \frac{\begin{pmatrix} 17 \\ 26 \end{pmatrix}\begin{pmatrix} 17 & 26 \end{pmatrix} + \begin{pmatrix} 25 \\ 19 \end{pmatrix}\begin{pmatrix} 25 & 19 \end{pmatrix}}{8}$$

$$= \begin{pmatrix} 4.2500 & 0.1250 \\ 0.1250 & 1.6250 \end{pmatrix}$$

Table 5: MANOVA Table for Hui-Dong and Ru-Gen Model

Source	df	SSP
Treatments	3	$\begin{pmatrix} 2.250 & -1.8750 \\ -1.8750 & 8.1875 \end{pmatrix}$
Rows	3	$\begin{pmatrix} 4.250 & -0.8750 \\ -0.8750 & 2.6875 \end{pmatrix}$
columns	3	$\begin{pmatrix} 10.250 & -5.6250 \\ -5.6250 & 6.1875 \end{pmatrix}$
Boxes	3	$\begin{pmatrix} 8.250 & -3.3750 \\ -3.3750 & 4.6875 \end{pmatrix}$
Error	3	$\begin{pmatrix} 0.7500 & 1.6250 \\ 1.6250 & 6.6875 \end{pmatrix}$
Total	15	$\begin{pmatrix} 25.750 & -10.1250 \\ -10.1250 & 28.4375 \end{pmatrix}$

Table 6: MANOVA Table of Sudoku design of Type I

Source	**df**	**SSP**
Treatments	3	$\begin{pmatrix} 2.250 & -1.8750 \\ -1.8750 & 8.1875 \end{pmatrix}$
Row blocks	1	$\begin{pmatrix} 0.250 & 0.6250 \\ 0.6250 & 1.5625 \end{pmatrix}$
Column block	1	$\begin{pmatrix} 4.000 & -3.500 \\ -3.500 & 3.0625 \end{pmatrix}$
Rows	3	$\begin{pmatrix} 4.250 & -0.8750 \\ -0.8750 & 2.6875 \end{pmatrix}$
columns	3	$\begin{pmatrix} 10.250 & -5.6250 \\ -5.6250 & 6.1875 \end{pmatrix}$
Boxes	3	$\begin{pmatrix} 8.250 & -3.3750 \\ -3.3750 & 4.6875 \end{pmatrix}$
Error	1	$\begin{pmatrix} -3.5000 & 5.7500 \\ 5.7500 & 2.0625 \end{pmatrix}$
Total	15	$\begin{pmatrix} 25.750 & -10.1250 \\ -10.1250 & 28.4375 \end{pmatrix}$

TABLE 7: MANOVA Table of Sudoku design of Type II

Source	df	SSP
Treatments	3	$\begin{pmatrix} 2.250 & -1.8750 \\ -1.8750 & 8.1875 \end{pmatrix}$
Row blocks	1	$\begin{pmatrix} 0.250 & 0.6250 \\ 0.6250 & 1.5625 \end{pmatrix}$
Column block	1	$\begin{pmatrix} 4.000 & -3.500 \\ -3.500 & 3.0625 \end{pmatrix}$
Rows withn row block	2	$\begin{pmatrix} 4.0000 & -1.5000 \\ -1.5000 & 1.1250 \end{pmatrix}$
columns withincolumnblock	2	$\begin{pmatrix} 6.2500 & -2.1250 \\ -2.1250 & 3.1250 \end{pmatrix}$
Boxes	3	$\begin{pmatrix} 8.250 & -3.3750 \\ -3.3750 & 4.6875 \end{pmatrix}$
Error	3	$\begin{pmatrix} 0.7500 & 1.6250 \\ 1.6250 & 6.6875 \end{pmatrix}$
Total	15	$\begin{pmatrix} 25.750 & -10.1250 \\ -10.1250 & 28.4375 \end{pmatrix}$

Table 8: MANOVA Table of Sudoku design of Type IV

Source	df	SSP
Treatments	3	$\begin{pmatrix} 2.250 & -1.8750 \\ -1.8750 & 8.1875 \end{pmatrix}$
Row blocks	1	$\begin{pmatrix} 0.250 & 0.6250 \\ 0.6250 & 1.5625 \end{pmatrix}$
Column block	1	$\begin{pmatrix} 4.000 & -3.500 \\ -3.500 & 3.0625 \end{pmatrix}$
Rows within row block	2	$\begin{pmatrix} 4.0000 & -1.5000 \\ -1.5000 & 1.1250 \end{pmatrix}$
Columnwithin column block	2	$\begin{pmatrix} 6.2500 & -2.1250 \\ 2.1250 & 3.1250 \end{pmatrix}$
Boxeswithin row block	2	$\begin{pmatrix} 8.0000 & -4.0000 \\ -4.0000 & 3.1250 \end{pmatrix}$
Boxes within column block	2	$\begin{pmatrix} 4.2500 & 0.1250 \\ 0.1250 & 1.6250 \end{pmatrix}$
Error	2	$\begin{pmatrix} -3.250 & 2.1250 \\ 2.1250 & 9.7500 \end{pmatrix}$
Total	15	$\begin{pmatrix} 25.750 & -10.1250 \\ -10.1250 & 28.4375 \end{pmatrix}$

Table 9: Wilk's Test for Hui-Dong and Ru-Gen Model

Source	df	λ_1	λ_2	Λ	α=0.05
Treatments	3	10.9116	0.5752	0.0539	0.00953
Row	3	13.6852	0.3279	0.0513	0.00953
Column	3	38.1625	0.3506	0.0189	0.00953
Boxes	3	29.3320	0.6282	0.0202	0.00953
Error	3				
Total	15				

Not Significant at α=0.05, null hypothesis not rejecte

Table 10: Wilk's Test for MANOVA Sudoku design of Type I

Source	df	λ_1	λ_2	Λ	$\Lambda(p, k-1, g)$
Treatments	3	0.6395	-0.5786	0.6099	0.0000
Row block	1	3.0141	-2.059e-6	0.2491	0.0000
Column block	1	-9.379e-1	2.6237e-17	0.9999	0.0000
Row	3	-0.6444	0.4105	0.700	0.0000
Column	3	-1.9896	0.3965	0.7161	0.0000
Boxes	3	-1.4467	0.4681	0.6811	0.0000
Error	1				
Total	15				

Not Significant at α=0.05, null hypothesis not rejected

Table 11: Wilk's test for MANOVA Sudoku design of Type II

Source	df	λ_1	λ_2	Λ	$\Lambda(p,k-1,g)$
Treatments	3	10.930	0.572	0.0534	0.00953
Row block	1	1.0000	0.2763	0.3918	0.0500
Column block	1	1.7022e+1	2.2855e-16	0.1777	0.0500
Row	2	13.6027	0.0697	0.0640	0.0180
Column	2	21.1972	0.2983	0.03470	0.0180
Boxes	3	28.9348	0.3971	0.0239	0.00953
Error	3				
Total		15			

Not Significant at α=0.05, null hypothesis not rejected

Table 12: Wilk's Test for MANOVA Sudoku design of Type IV

Source	df	λ_1	λ_2	Λ	$\Lambda(p,k-1,g)$
Treatments	3	-0.6882	0.5977	0.6259	0.00028
Row block	1	1.4631	-5.2583e-18	0.4060	0.00253
Column block	1	-1.2132e+00	9.8933e-19	0.9999	0.00253
Row s within block	2	-1.2073	0.2345	0.8100	0.00064
Columnswithin block	2	-1.8735	0.2245	0.8167	0.00064
Boxes within block	2	-2.4452	0.1017	0.9077	0.00064
Boxes within block	2	-1.1496	0.1656	0.8579	0.00064
Error	2				
Total	15				

Not Significant at α=0.05, null hypothesis not rejected

Table 13: Lawley-Hotelling Test for Hui-Dong and Ru-Gen Model

Source	df	λ_1	λ_2	U^s	$\frac{v_e U^s}{v_h}$	α=0.05
Treatments	3	10.9116	0.5752	11.5752	11.4868	58.915
Row	3	13.6852	0.3279	14.0131	14.0131	58.915
Column	3	38.1625	0.3506	38.5131	38.5131	58.915
Boxes	3	29.3320	0.6282	29.9605	29.9605	58.915
Error	3					
Total	15					

Not Significant at α=0.05, null hypothesis not rejected

Table 14: Lawley-Hotelling Test for MANOVA Sudoku design of Type I

Source	df	λ_1	λ_2	U^s	$\frac{v_e U^s}{v_h}$	α=0.05
Treatments	3	0.6395	-0.5786	0.6395	0.2132	0.0000
Row block	1	3.0141	-2.059e-6	3.0141	1.0047	0.0000
Columnblock	1	-9.379e-1	2.6237e-17	0.0000	0.0000	0.0000
Row	3	-0.6444	0.4105	0.4105	0.1368	0.0000
Column	3	-1.9896	0.3965	0.3965	0.1322	0.0000
Boxes	3	-1.4467	0.4681	0.4681	0.1560	0.0000
Error	1					
Total	15					

Not Significant at α=0.05, null hypothesis not rejected

Table 15: Lawley-Hotelling test for MANOVA Sudoku design of Type II

Source	df	λ_1	λ_2	U^s	$\frac{v_e U^s}{v_h}$	α=0.05
Treatments	3	10.930	0.572	11.4882	11.4882	58.915
Row block	1	1.0000	0.2763	1.2763	3.8289	58.428
Column block	1	1.7022e+1	2.2855e-16	17.022	51.0660	58.428
Row within Boxes	2	13.6027	0.0697	13.6724	20.5086	58.915
Column within Boxes	2	21.1972	0.2983	21.4955	32.2433	58.915
Boxes	3	28.9348	0.3971	29.3319	29.3319	58.915
Error	3					
Total	15					

Not Significant at α=0.05, null hypothesis not rejected

Table 16: Lawley-Hotelling test for MANOVA Sudoku design of Type IV

Source	df	λ_1	λ_2	U^s	$\frac{v_e U^s}{v_h}$	α=0.05
Treatments	3	-0.6882	0.5977	0.5977	0.3985	10.659
Row block	1	1.4631	-5.2583e-18	1.4631	2.9262	0.0000
Column block	1	-1.2132e+00	9.8933e-19	9.8933e-19	0.0000	0.0000
Row s within block	2	-1.2073	0.2345	0.2345	0.2345	9.8591
Columnswithin block	2	-1.8735	0.2245	0.2245	0.2245	9.8591
Boxes within block	2	-2.4452	0.1017	0.1017	0.1017	9.8591
Boxes within block	2	-1.1496	0.1656	0.1659	0.1659	9.8591
Error	2					
Total	15					

Not Significant at α=0.05, null hypothesis not rejected

Table 17: Roy's test for Hui-Dong and Ru-Gen Model

Source	df	λ_1	λ_2	θ	$\theta(\alpha, s, q, v)$
Treatments	3	10.9116	0.5752	0.9164	0.5650
Row	3	13.6852	0.3279	0.9319	0.5650
Column	3	38.1625	0.3506	0.9745	0.5650
Boxes	3	29.3320	0.6282	0.9670	0.5650
Error	3				
Total	15				

Significant at α=0.05, null hypothesis is rejected.

Table 18: Roy's test for MANOVA Sudoku design of Type I

Source	df	λ_1	λ_2	θ	$\theta(\alpha, s, q, v)$
Treatments	3	0.6395	-0.5786	0.3901	0.5650
Row block	1	3.0141	-2.059e-6	0.7509	0.5650
Columnblock	1	-9.379e-1	2.6237e-17	0.0000	0.5650
Row	3	-0.6444	0.4105	0.2910	0.5650
Column	3	-1.9896	0.3965	0.2839	0.5650
Boxes	3	-1.4467	0.4681	0.3188	0.5650
Error	1				
Total	15				

Not significant at α=0.05, null hypothesis not rejected

Table 19: Roy's test for MANOVA Sudoku design of Type II

Source	df	λ_1	λ_2	θ	$\theta(\alpha, s, q, v)$
Treatments	3	10.930	0.572	0.9161	0.5650
Row block	1	1.0000	0.2763	0.5000	0.5650
Column block	1	1.7022e+1	2.2855e-16	0.9445	0.5650
Row within Boxes	2	13.6027	0.0697	0.9315	0.5650
Column within Boxes	2	21.1972	0.2983	0.9545	0.5650
Boxes	3	28.9348	0.3971	0.9666	0.5650
Error	3				
Total	15				

Significant at α=0.05, null hypothesis is rejected

Table 20: Roy's test for MANOVA Sudoku design of Type IV

Source	df	λ_1	λ_2	θ	$\theta(\alpha, s, q, v)$
Treatments	3	-0.6882	0.5977	0.3741	0.5650
Row block	1	1.4631	-5.2583e-18	0.5940	0.5650
Column block	1	-1.2132e+00	9.8933e-19	0.0000	0.5650
Row s within block	2	-1.2073	0.2345	0.1899	0.5650
Columnswithin block	2	-1.8735	0.2245	0.817	0.5650
Boxes within block	2	-2.4452	0.1017	0.0923	0.5650
Boxes within block	2	-1.1496	0.1656	0.1421	0.5650
Error	2				
Total	15				

Not Significant at α=0.05, null hypothesis not rejected

Table 21: Pillai test for Hui-Dong and Ru-Gen Model

Source	df	λ_1	λ_2	V^s	$\theta(\alpha, s, q, v)$
Treatments	3	10.9116	0.5752	1.2812	1.536
Row	3	13.6852	0.3279	1.1788	1.536
Column	3	38.1625	0.3506	1.2341	1.536
Boxes	3	29.3320	0.6282	1.3530	1.536
Error	3				
Total	15				

Not Significant at α=0.05, null hypothesis not rejected

Table 22: Pillai test for MANOVA Sudoku design of Type I

Source	df	λ_1	λ_2	V^s	$\theta(\alpha, s, q, v)$
Treatments	3	0.6395	-0.5786	0.3901	1.536
Row block	1	3.0141	-2.059e-6	0.7509	1.536
Columnblock	1	-9.379e-1	2.6237e-17	0.0000	1.536
Row	3	-0.6444	0.4105	0.2910	1.536
Column	3	-1.9896	0.3965	0.2839	1.536
Boxes	3	-1.4467	0.4681	0.3188	1.536
Error	1				
Total	15				

Not Significant at α=0.05, null hypothesis not rejected

Table 23: Pillai test for MANOVA Sudoku design of Type II

Source	df	λ_1	λ_2	V^s	$\theta(\alpha, s, q, v)$
Treatments	3	10.930	0.572	1.2811	1.536
Row block	1	1.0000	0.2763	0.7189	1.536
olumn block	1	1.7022e+1	2.2855e-16	0.8223	1.536
Row within Boxes	2	13.6027	0.0697	0.9967	1.536
Column within Boxes	2	21.1972	0.2983	1.1770	1.536
Boxes	3	28.9348	0.3971	1.2508	1.536
Error	3				
Total	15				

Not Significant at α=0.05, null hypothesis not rejected

Table 24: Pillai test for MANOVA Sudoku design of Type IV

Source	df	λ_1	λ_2	V^s	$\theta(\alpha, s, q, v)$
Treatments	3	-0.6882	0.5977	0.3741	1.536
Row block	1	1.4631	-5.2583e-18	0.5940	1.536
Column block	1	-1.2132e+00	9.8933e-19	0.0000	1.536
Row s within block	2	-1.2073	0.2345	0.1899	1.536
Columnswithin block	2	-1.8735	0.2245	0.1833	1.536
Boxes within block	2	-2.4452	0.1017	0.0923	1.536
Boxes within block	2	-1.1496	0.1656	0.1420	1.536
Error	2				
Total	15				

Not significant at α=0.05, null hypothesis not rejected

Discussion

This research paper aimed at modifying existing univariate Sudoku square design models to a multivariate case. However, the study proposed procedures of obtaining multivariate analysis of variance for the models and also procedures for calculating the sums of square and products using numerical example. The hypothetical data used for this study, assumed to have multivariate normal with p-vector mean and constant variance-covariance matrix (sigma). The SSP on the MANOVA tables contain matrices of order 2, because sample size of the data used is 2. The main diagonal is the sum of squares while off the diagonal is the sum of products. The results of significant tests show that Wilk's Lambda, Pillai and Lawley-Hotteling, test no significant at α=0.05 for all the effects in all the models proposed. However, Roy,s largest root showed significant effect for all effects in the Hui-Dong and Ru-Gen model and model II of Subramani and Ponnuswamy.

It was observed that three of the univariate Sudoku models suggested by Subramani and Ponnuswamy (2009) were modified in this research, Type III was not modified. The reason is, owing to the fact that the available hypothetical data was insufficient to carry out MANOVA for the model.

Conclusion

This paper modified the existing univariate Sudoku square models suggested by Hui-Dong and Ru-Gen (2008) and Subramani and Ponnuswamy (2009) to cater for more than one dependent variables. It also revealed the procedures for manual computation of sum of squares and products of Sudoku square design when the dependent variables are more than one (multivariate).The generalized linear model for Sudoku square models was proposed and MANOVA tests of significance were carried out using four different tests at α=0.05.

This paper recommended that higher order of Sudoku square design should be used to avoid some models failing due to insufficient data, as regarding the case of model III, suggested by Subramani and Ponnuswamy (2009) that failed under Sudoku square of order 4. Further, the use of four multivariate tests to test significance of effects should be encouraged for these proposed multivariate Sudoku models as each of these tests has the same probability of rejecting null hypothesis and however, at times in a given data set conclusions might be differed even when null hypothesis is true.

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