CONVERGENCE OF REFINED S-ITERATION IN BANACH SPACE

RAUF, K.[1]; KANU, R. U.[2]; IBRAHIM, L. A.[3]; WAHAB, O. T.[4]; & FATAI, M. O.[5]
[1, 3 & 5]Department of Mathematics, University of Ilorin, Ilorin, Nigeria
[2]Department of Basic Sciences (Mathematics Unit),
Babcock University, Ilishan- Remo, Ogun State, Nigeria.
[4]Department of Statistics and Mathematical Sciences,
Kwara State University, Malete, Nigeria.
E-mail: krauf@unilorin.edu.ng; richmondkanu2004@yahoo.com; lukmanibrah@gmail.com; taofeek.wahab@kwasu.edu.ng; musakalamullah@gmail.com
Phone No: 234-814-641-6645

Abstract
The convergence of a sequence generated by a refined S-iteration scheme for finding a common fixed point of non-expansive mappings in a Banach space are obtained. Numerical example to validate our results is generated.

Keywords: *Convergence, Iterative Scheme, Banach Space, Fixed point & Operator.*

Introduction

There are several fixed point iteration methods and several fixed point theorems underlying them. One of the earliest uses was picard iteration method for proving existence of solutions of Ordinary Differential Equation. It is based on the "Banach fixed point theorem" though Banach was not born yet when Picard discovered it.

The origin of fixed point theory lies in the method of successive approximation used for proving existence of solutions of differential equations introduced independently by Joseph Liouville in 1837 and Charles Emile Picard in 1890 (see Mann; 1953; Xue & Zhang, 2013). But formally it was started in the beginning of twentieth century as an important part of analysis. The abstraction of this classical is the pioneering work of the great Polish Mathematician Stefan Banach published in 1922 which provides a constructive method to find the fixed point of a map. Fixed point iterative procedure are designed to be applied in solving equations arising in physical formulation but there is no systematic study of numerical aspects of these iterative procedures. In computational Mathematics, it is of interest to know which of the given iterative procedures converges faster to a desire solution, commonly known as rate of convergence. Rhoades (1995) compared the Mann and Ishikawa iterative procedures concerning their rate of convergence. He illustrated the difference in the rate of convergence for increasing and decreasing functions. Indeed, he used computer programs, perhaps for the first time to compare the Mann and Ishikawa iteration through examples.

Iterative methods are often the only choice for non linear equations. however, iterative methods are often useful even for linear problems involving a large number of variables where direct method would be prohibitively expensive even with the best available computing power. However the iterative scheme adopted in this paper work are Ishikawa, Picard-Mann, S-iteration and Refined S-iteration [see Zamfirescu, 1972), Liu, 1995; Rauf et. al. (2017)].

The aim of this paper is to establish the strong convergence, convergence rate of iterative schemes with errors using Zamfirescu operator in Banach spaces. The main objectives are to find a common iterative scheme for some iterative algorithms, obtain the uniqueness of the iterative schemes.

In particular, we adjusted Picard-Mann iteration in form of S-iteration and prove the strong convergence of a sequence generated by the adapted S-iteration scheme with errors in a Banach space. Numerical example is used to validate the results.

Some Special Iterations
Picard Iteration
The Picard iteration process is defined by the sequence

$$x_{n+1} = Tx_n, \qquad n \geq 0 \tag{2.1}$$

and the concept of Picard iteration process with error is defined as follows

$$x_{n+1} = Tx_n + u_n, \qquad n \geq 0 \tag{2.2}$$

where u_n satisfy

$$\sum_{n=1}^{\infty} \| u_n \| < \infty \tag{2.3}$$

Picard-Mann Iteration
Picard-Mann iteration is given by

$$x_{n+1} = Ty_n$$
$$y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \qquad n \geq 0 \tag{2.4}$$

where α_n is a sequence in $[0,1]$ with $\sum \alpha_n = \infty$

while the Picard-Mann iteration process with errors is given as

$$x_{n+1} = Ty_n + e_n$$

,

$$y_n = (1-\alpha_n)x_n + \alpha_n Tx_n + f_n, \qquad n \geq 0 \tag{2.5}$$

where e_n and f_n satisfy

$$\sum_{n=1}^{\infty} \| f_n \| < \infty$$

and

$$\sum_{n=1}^{\infty} \| \infty \| < \infty$$

Ishikawa Iteration
Ishikawa iteration defined another iteration process given by the sequence

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n Tx_n$$

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n, \qquad n \geq 0 \tag{2.6}$$

where α_n and β_n are sequence in $[0,1]$

The Ishikawa iteration is a double Mann iteration and has better approximation than Mann iteration.

The Ishikawa iteration with errors is given as:

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n Tx_n + e_n$$

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n + f_n, \qquad n \geq 0 \tag{2.7}$$

where e_n and f_n satisfy

$\sum_{n=1}^{\infty} \| e_n \| < \infty$ and $\sum_{n=1}^{\infty} \| f_n \| < \infty$ respectively.

S-Iteration

In attempt to reduce computational cost, Agarwal defined another iteration called S-iteration which is independent of Mann and Ishikawa iteration and it is defined by the sequence:

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n$$

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n, \qquad n \geq 0 \tag{2.8}$$

where α_n and β_n are sequence in $[0,1]$.

The S-iteration with errors can be given as

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n + e_n$$

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n + f_n, n \geq 0 \tag{2.9}$$

where e_n and f_n satisfy

$$\sum_{n=1}^{\infty} \| f_n \| < \infty$$

and

$$\sum_{n=1}^{\infty} \| \infty \| < \infty$$

See, Agarwa et. al (2007).

Some Important Terms

Fixed point: Fixed point of a function is an element of the the function's domain that is mapped to itself by the function. that is to say, x is a fixed point of the function $f(x)$ if and only if $f(x) = x$.

Fixed point iteration: Given a function f defined on the real numbers with real values and given a point x_0 in the domain of f, the fixed point iteration is

$$x_{n+1} = f(x_n)$$

, $n = 0,1,2....$ which gives rise to the sequence x_0, x_1, x_2 $\cdots$ which is hoped to converge to a point x

Contraction mapping: A contraction mapping on a metric space (M, d) is a function f from M to itself, with the property that there is some non-negative real number $0 \leq k < 1$ such that for all x and y in M,

$$d(f(x), f(y)) \leq kd(x, y)$$

The smallest value of k is called the Lipschitz constant of f. Contractive maps are sometimes called lipschitizian maps. if the above condition is instead satisfied for $k \leq 1$, then the mapping is said to be a non-expansive map.

Main Results

Banach proved the convergence of Picard iteration process

$$x_{n+1} = Tx_n, \qquad n \geq 0$$

with the aid of the following contractive mapping.

$$d(Tx, Ty) \leq ad(x, y) \tag{3.1}$$

where $T : X \to X$. (X, d) is a metric space, $a \in (0,1)$ and $x, y \in X$. Banach's theorem is given as follow:

Theorem 3.1: *Let (X, d) be a metric space and $T : X \to X$ be a contraction map on X. Then T has a unique fixed point $x \in X$.*

When condition of the contraction mapping is weaker, Picard iteration will no longer converge to a fixed point. Hence, other iteration such as Mann iteration, Picard-Mann iteration, Ishikawa iteration, Thiawan iteration and S-iteration would be considered.

The most generalized operator used to approximate fixed point is the one proved by Zamfirescu. The Zamfirescu operator was obtain from the Banach, Kannan and Chatterjea contractive mappings [See, Kannan (1968)] as follows: The operator T is called a Kannan mapping if there exist $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \quad \forall\ x,y \in X \tag{3.2}$$

Another similar definition due to Chatterjea mapping is as follows:

there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \ \forall\ x,y \in X. \tag{3.3}$$

By combining (3.1), (3.2) and (3.3) conditions we have the Zamfirescu operator given, for $x, y \in X$ by

$$Z_1 : d(Tx, Ty) \leq ad(x, y), \quad \forall a \in (0,1)$$

$$Z_2 : d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \qquad b \in (0, \frac{1}{2})$$

$$Z_3 : d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)];\ c \in (0, \frac{1}{2}) \tag{3.4}$$

The equivalence of (3.4) is given as follows:

$$d(Tx, Ty) \leq h\max[d(x,y), d(x, Tx), d(y,Ty), d(x, Ty), d(y, Tx)] \tag{3.5}$$

where

$$h = max(a, b, c) \qquad and \qquad x, y \in X$$

The following Lemma is useful in the proof of our results.

Lemma 3.1
Let δ be real number such that $\delta \in [0,1)$ and $u_{n=0}^{\infty}$ is a sequence of non negative such that:

$$\lim_{n\to\infty} u_n = 0$$

Then for any sequence of positive numbers $x_{n=0}^{\infty}$ satisfying

$$x_{n+1} \leq \delta x_n + u_n \quad \forall \ n \in N \tag{3.6}$$

we have

$$\lim_{n\to\infty} x_n = 0 \tag{3.7}$$

The Zamfirescu operator was used to proved the strong convergence of Picard iteration process is as stated below:

Theorem 3.2 *Let C be a nonempty subset of a normed space $(E, \|.\|)$. Let $C : \to C$ be Z-operator. if $F(T) \neq \phi$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\| U_n \| = 0$. Then x_n converges strongly to a fixed point of T.*

Proof
By Lemma 3.1, T has a unique fixed point in C, say q, let $x, y \in C$; since T is a Z-operator, at least one of each condition Z_1 to Z_3 is satisfied. if Z_2 holds, then

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \tag{3.8}$$

is equivalent to

$$\| Tx - Ty \| \leq b(\| x - Tx \| + \| y - Ty \|)$$

$$\leq b[\|x-Tx\| + \|y-x\| + \|x-Tx\| + \|Tx-Ty\| \tag{3.9}$$

this implies

$$(1-b)\|Tx-Ty\| \leq b\|x-y\| + 2b\|x-Tx\| \tag{3.10}$$

therefore

$$\|Tx-Ty\| \leq \frac{b}{1-b} \|x-y\| + \frac{2b}{1-b} \|x-Tx\| \tag{3.11}$$

Similarly, if Z_3 holds, we obtain

$$\|Tx-Ty\| \leq \frac{c}{1-c} \|x-y\| + \frac{2c}{1-c} \|x-Tx\| \tag{3.12}$$

Let

$$\delta = max(a, \frac{b}{1-b}, \frac{c}{1-c}) \tag{3.13}$$

Then, we have $0 \leq \delta \leq 1$ and in view of (3.13) it results in inequalities (3.11) and (3.12), hence,

$$\|Tx-Ty\| \leq \delta \|x-y\| + 2\delta \|x-Tx\| \ \forall \ x,y \ C \tag{3.14}$$

Now, for $x = q$ and $y = x_n$ in (3.14), we obtain

$$\|Tq-Tx_n\| \leq \delta \|q-x_n\| \tag{3.15}$$

which implies

$$\|x_{n+1} - q\| \leq \delta \|x_n - q\| \tag{3.16}$$

By Lemma 3.1, we conclude that x_n converges strongly to q.
The adjusted S-iteration with errors:

$$x_n = (1-\alpha_n)Ty_{n-1} + \alpha_n Ty_{n-1} + e_n$$
$$y_{n-1} = (1-\beta_n)x_{n-1} + \beta Tx_{n-1} + f_n$$

We shall show the existence of the restructured S-iteration from the theorem below:

Theorem 3.3 *Let* C *be a nonempty subset of a normed space* $(E, \|.\|)$ *. Let* $T : C \to C$ *be Z-operator. let* x_n *be defined with Modified S-iterative process. If* $F(T) \neq \phi$ *,* $\sum_{n=1}^{\infty} \infty_n = \infty$ *,* $\| U_n \| = 0$ *. Then* x_n *converges to a fixed point of* T *.*

Proof

from (3.14), we have

$$\| Tx - Ty \| \leq \delta \| x - y \| + 2\delta \| x - Tx \| \qquad \forall x, y \in C$$

Also from (3.16), we have,

$$\| Tx_n - q \| \leq \delta \| x_n - q \|$$

Applying (3.16) to the above Modified S-iteration, we obtain

$$\begin{aligned}
\| x_n - q \| &\leq \| (1-\alpha_n) T y_{n-1} + \alpha_n T y_{n-1} + e_n - q \| \\
&= \| (1-\alpha_n)(T y_{n-1} - q) + \alpha_n (T y_{n-1} - q) + e_n \| \\
&\leq \| (1-\alpha_n)(T y_{n-1} - q) \| + \| \alpha_n (T y_{n-1} - q \| + \| e_n \| \\
&= (1-\alpha_n) \| T y_{n-1} - q \| + \alpha_n \| T y_{n-1} - q \| + \| e_n \| \\
&\leq (1-\alpha_n) \delta \| y_{n-1} - q \| + \alpha_n \delta \| y_{n-1} - q \| + \| e_n \| \\
&= [(1-\alpha_n) \delta + \alpha_n \delta] \| y_{n-1} - q \| + \| e_n \| \quad (3.17)
\end{aligned}$$

Also by the same argument, we have

$$\begin{aligned}
\| y_{n-1} - q \| &\leq \| (1-\beta_n) x_{n-1} + \beta_n T x_{n-1} + f_n - q \| \\
&= \| (1-\beta_n) x_{n-1} + \beta_n T x_{n-1} - q + f_n - \beta_n q + \beta_n q \| \\
&= \| (1-\beta_n)(x_{n-1} - q) + \beta_n (T x_{n-1} - q) + f_n \| \\
&\leq (1-\beta_n) \| x_{n-1} - q \| + \beta_n \delta \| x_{n-1} - q \| + \| f_n \| \\
&= [(1-\beta_n) + \beta_n \delta \| x_{n-1} - q \| + \| f_n \| \qquad (3.18)
\end{aligned}$$

substituting (3.17) in Theorem (3.3)

$$\begin{aligned}
\| x_n - q \| &\leq [(1-\alpha_n)\delta + \alpha_n \delta][(1-\beta_n) + \beta_n \delta] \| x_{n-1} - q \| + \| e_n \| + \| f_n \| \\
&= (1-\alpha_n + \alpha_n) \delta (1-\beta_n + \beta_n \delta) \\
&= \delta (1-(1-\delta) \beta_n \| x_{n-1} - q \| + \| e_n \| + \| f_n \| \qquad (3.19) \\
&(1-(1-\delta) \beta_n) \| x_n - q \| + \| e_n \| + \| f_n \| \quad (3.20)
\end{aligned}$$

By letting $\| g_n \| = \| e_n \| + \| f_n \|$, and by lemma 3.1, using the condition of the theorem, we have

$$\lim_{n\to\infty} \| x_n - q \| = 0$$

Hence

$$x_n \to q \in F(T)$$

Numerical Examples

We shall support our analytical results with the following Numerical examples using MATHLAB and compare the rate of convergence of Ishikawa, S-iteration and the Adapted S-iteration.

Example

Let the function $f:[0,4]\rightarrow[0,4]$ be defined by $f(x)=(2x+3)^{1/2}$ with fixed point $q=3.0000$. initial guess is $x_0=4$, $\alpha_n=\frac{1}{(n+2)}$ $\beta_n=\frac{1}{(n+1)}$

Numerical Results for Example 1

n	Ishikawa	Picard-mann	S-iteration	Adapted S-iteration
	4.000000000	4.000000000	4.000000000	4.000000000
	3.509525033	3.247216735	3.281733296	3.211950309
	3.328291361	3.067813070	3.087368905	3.054367130
	3.236413828	3.019521294	3.028018410	3.015059648
	3.181820429	3.005778131	3.009117873	3.004347135
	3.146067652	3.001740752	3.002989544	3.001287744
	3.121052680	3.000532251	3.000984449	3.000388340
	3.102689324	3.000164270	3.000325106	3.000118657
	3.088707277	3.0000511059	3.000107564	3.000036622
	3.077750686	3.000016002	3.000035372	3.000011393
	3.068963134	3.000005037	3.000011819	3.000003567
	3.061778846	3.000001593	3.000003922	3.000001123
	3.055810057	3.000000505	3.000001302	3.000000357
	3.050782729	3.0000000161	3.000000432	3.000000112
	3.046498066	3.000000051	3.000000143	3.000000035
	3.042808626	3.0000000016	3.000000048	3.000000014
	3.039602883	3.000000005	3.000000016	3.000000003
	3.036795039	3.000000001	3.000000005	3.000000001

	3.034180862	3.000000000	3.000000001	3.000000000
	3.032118983	3.000000000	3.000000000	3.000000000
	3.030155239	3.000000000	3.000000000	3.000000000
	3.028392438	3.000000000	3.000000000	3.000000000
	3.026802437	3.000000000	3.000000000	3.000000000
	3.025362007	3.000000000	3.000000000	3.000000000

n	Ishikawa	Picard-mann	S-iteration	Adapted S-iteration
	3.024518254	3.000000000	3.000000000	3.000000000
	3.022855690	3.000000000	3.000000000	3.000000000
	3.021759929	3.000000000	3.000000000	3.000000000
	3.019824750	3.000000000	3.000000000	3.000000000
	3.018966875	3.000000000	3.000000000	3.000000000
	3.018171931	3.000000000	3.000000000	3.000000000
	3.000867436	3.000000000	3.000000000	3.000000000

Conclusion

The convergence and convergence rate of a two-step iterative schemes with error using Zamfirescu operator in Banach spaces were proved. It was observed from the example considered, that the adapted S-iteration converges faster than the S and Ishikawa iterations. However, the scheme has the same convergence rate with Picard-Mann iteration. The adapted S-iteration is recommended for solving linear and quadratic problems involving a large number of variables where direct method would be prohibitively expensive even with the best available computing power. The result obtained from the numerical computation also confirm the validity of theoretical analysis.

References

Agarwa, R. P., O'Regan, D., & Sahu, D. R. (2007). Iterative construction of fixed points of nearly asympotically non-expansive mappings. *Non-linear Convex Anal.,* 8(1), 67-79.

Banach. S. (1932). *Theories des operation lineires.* Warszawa, Poland: Warsaw Publishing House.

Berinde, V. (2007). *Iterative approximation of fixed points*. New York: Springer Berlin Heidelberg.

Imoru, C. O., & Olantiwo, M. O. (2003). On the stability of Picard and Mann iteration processes. On the stability of Picard and Mann iteration Processes, 19, 155-160.

Ishikawa, S. (1974). *Fixed points by a new iteration method.* Proceedings of the American Mathematical Society, 44(1), 147-150.

Kannan, R. (1968). Some results on fixed points. *Bulletines of the Calcutta Mathematical Society,* 60, 71-76.

Liu, L. S. (1995). Ishikawa and Mann iterative process with errors for non-linear strongly accretive mappings in Banach Spaces. *J. Math. Anal. appl*., 194(1), 114-125.

Mann, W. R. (1953). Mean value methods in iterations. *Proc. Amer. Math. Soc.;* 4, 506-510.

Rauf, K., Wahab, O. T., & Ali, A. (2017). New implicit kirk-type schemes for general class of quasi-contractive operators in generalized convex metric spaces. *The Australian Journal of Mathematical Analysis and Applications(AJMAA),* 14(1)(8), 1-29.

Rhoades, B. E. (1995). A general principle for Mann iterates. *Indian J. Pure Appl. Math.*, 26(8), 751-762.

Xue, Z., & Zhang, H. (2013). The convergence of implicit Mann and Ishikawa iteration for weak generalized $varphi$ -hemicontractive mapping in real Banach spaces. *Journal of Inequalities and Applications,* article 231, 1-11.

Zamfirescu, T. (1972). *Fix point theorems in metric spaces. Archiv der Mathematik*, 23(1), 292-298.