A NOTE ON THE PROPERTIES OF SOLUTION OF TRANSIENT HYDROMAGNETIC FREE CONVECTIVE FLOW PAST AN INFINITE VERTICAL POROUS PLATE

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Abstract
This work examines the properties of solution of two dimensional flow of a viscous incompressible electrically conducting fluid past an infinite vertical porous plate in a porous medium in the presence of uniform transverse magnetic field and constant heat source. Our results revealed that velocity u *, mass* ϕ *and temperature* θ *are increasing function of time.*

Keywords: Convective flow, porous medium, porous plate, conducting fluid and heat

Introduction
Convective flows are important in the context of process involving high temperatures. In many engineering areas such as nuclear power plants, gas, turbines and various propulsion devices for aircraft, missiles and space vehicles. The effect of free convection on accelerated flow of a viscous incompressible fluid past an infinite vertical plate with suction has many important technological applications in the astrophysical, geophysical and engineering problems .The study of the flow of an electrically conducting fluid over porous media has been studied due to its numerous applications such applications include MHD pumps, induction pumps, MHD generators , oil exploration, nuclear power plants ,gas turbines, air crafts and space vehicles among many others. Seigel (1958) first studied transient free convection flow past a semi-infinite vertical plate by an integral method. Since then many researchers have been published papers on free convection flow past a semi-infinite vertical plate.

A few other works of interest in this area include the works of Ogulu and Prakash (2006), Kim (2000), Makinde (2005) and Ogulu and Makinde (2009). Anand *et al*. (2012) used finite element method (FEM) to obtain the solution of heat and mass transfer in MHD flow of a viscous fluid past a vertical plate under oscillatory suction velocity. Sharma *et al*. (2012) investigated the flow of a viscous incompressible electrically conducting fluid along a porous vertical isothermal non-conducting plate with variable suction and internal heat generation in the presence of transverse magnetic field. Mohammed *et al*. (2015) presented an analytical method to describe the heat and mass transfer in the flow of an incompressible viscous fluid past an infinite vertical plate. With the governing equations accounting for the viscous dissipation effect and mass transfer with chemical reaction of constant reaction rate. The couple differential equations were transformed using similarity transformation and solved analytically using iteration perturbation method. Hamad *et al*. (2011) investigated the unsteady magneto hydrodynamic flow of a Nano fluid past an oscillatory moving vertical permeable semi-infinite flat plate with constant heat source in a rotating frame of reference.

The velocity along the plate (slip velocity) is assumed to oscillate on time with a constant frequency. Das and Jana (2010) investigated the effect of heat and mass transfer on the unsteady free convection flow of a viscous, electrically conducting incompressible fluid near an infinite vertical plate embedded in porous medium which moves with time dependent velocity under the influence of uniform magnetic field applied normal to the plate. An exact solution of the governing partial differential equation is obtained by using Laplace transform technique. Maina *et al*. (2015) studied the effects of heat transfer on unsteady MHD free convective flow past a vertical porous plate in a porous medium with heat source and constant injection. Crank-Nicolson method (FDM) was used to solve the governing coupled differential equations.

The objectives of this paper are to establish the criteria for the existence of unique solution of two dimensional flow of a viscous incompressible electrically conducting fluid past an infinite vertical porous plate in a porous medium in the presence of uniform transverse magnetic field and constant heat source and examine the properties of the solution under certain conditions.

Model Formulation

Consider the two dimensional flow of a viscous incompressible electrically conducting fluid past an infinite vertical porous plate in a porous medium in the presence of uniform transverse magnetic field (B_0) and constant heat source (Q). The x-axis is measured along vertical plate and y-axis normal to it as shown figure 1. The surface of the vertical plate is at uniform temperature T and concentration C. The temperature and concentration far away from the plate are T_∞ *and* C_∞ respectively. A magnetic field of strength B_0 acts normal to the plate that is, along the y-axis. The analysis of this study is based on following assumptions:

(i) Physical properties are assumed as constant.
(ii) Fluid particles are assumed as electrically conducting.

The physical sketch and geometry of the problem is shown in figure 1:

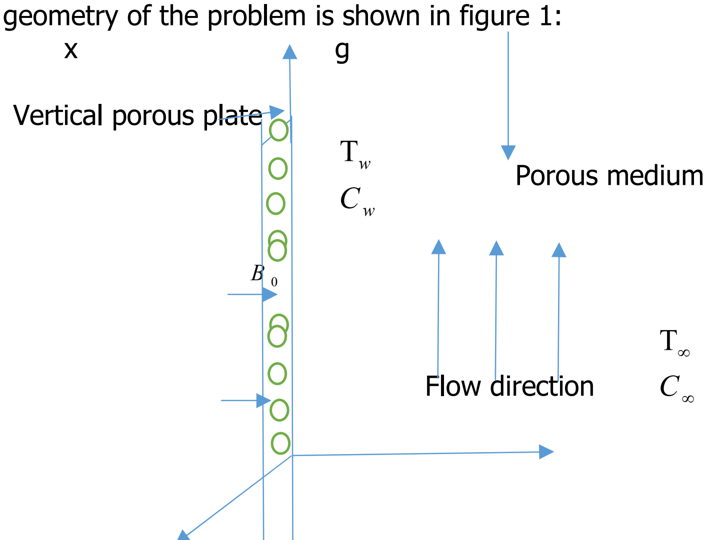

Figure 1: The flow configuration

Using these assumptions together with usual boundary layer approximations and following Maina *et al*. (2015) and Mohammed *et al*. (2015) we get the two dimensional equations describing the phenomenon as:

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

(1) **Momentum equation**

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\upsilon}{K}u-\frac{\sigma B_0^2}{\rho}u+g\beta'(C-C_\infty)\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\upsilon}{K}v-\frac{\sigma B_0^2}{\rho}v+g\beta'(C-C_\infty)\end{aligned}\right\} \tag{2}$$

Energy equation

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)-\frac{\upsilon}{c_p}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)^2+Q(T-T_\infty) \tag{3}$$

The equation for species concentrations

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_m\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right) \tag{4}$$

Where u, v are the dimensionless velocity components along the $x-$ and $y-$ directions respectively, υ is the kinematic viscosity, k thermal conductivity, σ is the electrical conductivity, B_0 the constant applied magnetic field, ρ the fluid density, g gravity acceleration, β^1 the concentration expansion coefficient, C and C_∞ are the concentration of solute at the plate and far away from the plate respectively. T is the temperature of the fluid on the surface of the plate, T_∞ the temperature of fluid far away from the plate, c_p is the specific heat capacity at constant pressure, Q additional heat source, and D_m is the molecular diffusivity.

The problem is two-dimensional and since the plate is an infinite, the velocity vector
$\bar{q}=(\mathrm{u},0), \quad \mathrm{u}=\mathrm{u}(\mathrm{x},\mathrm{y},\mathrm{t}) \;, \; v=v(\mathrm{x},\mathrm{y},\mathrm{t})$

By symmetry and from continuity equation (1)
$u=u(y,t)$, $T=T(y,t)$ and $C=C(y,t)$
Then, equations (1) – (4) reduce to

$$\frac{\partial u}{\partial t}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\upsilon}{K}u-\frac{\sigma B_0^2}{\rho}u+g\beta'(C-C_\infty) \tag{5}$$

$$\frac{\partial T}{\partial t}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}-\frac{\upsilon}{c_p}\left(\frac{\partial u}{\partial y}\right)^2+Q(T-T_\infty) \tag{6}$$

$$\frac{\partial C}{\partial t}=D_m\frac{\partial^2 C}{\partial y^2} \tag{7}$$

With initial and boundary conditions

$$\left.\begin{array}{lll}\mathrm{u}(\mathrm{y},0)=U_\infty & u(0,t)=U_w & u(\infty,t)=0\\ T(\mathrm{y},0)=\mathrm{T}_\infty & T(0,\mathrm{t})=T_w & T(\infty,t)=T_\infty\\ C(y,0)=\mathrm{C}_\infty & C(0,t)=C_w & C(\infty,t)=C_\infty\end{array}\right\} \tag{8}$$

Method of Solution

Non-dimensionalisation

We introduce dimensionless variables for space and time,

$$t' = \frac{D_m t}{L^2}, \qquad y' = \frac{y}{L} \tag{9}$$

We also introduce dimensionless variables for velocity, temperature and concentration;

$$u' = \frac{u}{U_\infty}, \qquad \theta = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \phi = \frac{C - C_\infty}{C_w - C_\infty}, \tag{10}$$

Using (9) and (10), and after dropping the prime, the equations (5) - (8) become

$$\frac{\partial u}{\partial t} = Sc\frac{\partial^2 u}{\partial y^2} - \left(M + \frac{1}{K_p}\right)u + Gr_\phi \phi \tag{11}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2} + Ec\left(\frac{\partial u}{\partial y}\right)^2 + q\theta \tag{12}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial y^2} \tag{13}$$

With initial and boundary conditions

$$\left.\begin{array}{lll} u(y,0) = 1, & u(0,t) = \alpha, & u(\infty,t) = 0 \\ \theta(y,0) = 0, & \theta(0,t) = 1, & \theta(\infty,t) = 0 \\ \phi(y,0) = 0, & \phi(0,t) = 1, & \phi(\infty,t) = 0 \end{array}\right\} \tag{14}$$

Where

$M = \dfrac{\sigma B_0^2 L^2}{D_m \rho}$ magnetic parameter, $K_p = \dfrac{kD_m}{\upsilon L^2}$ *permeability parameter*, $Sc = \dfrac{\upsilon}{D_m}$ Schmidt number,

$Gr_\phi = \dfrac{g\beta' L^2 (C_w - C_\infty)}{U_\infty D_m}$ *Grashof number*,

$\Pr = \dfrac{\rho C_p D_m}{K}$ Prandtl number, $Ec = \dfrac{U_\infty^2 \upsilon}{C_p\left(T_w - T_\infty\right) D_m}$ Eckert number,

$q = \dfrac{QL^2}{D_m}$ *heat source parameter.*

Existence and Uniqueness of Solution

Here, we shall prove the existence and uniqueness of solution of system of equation (11) - (13). The question of existence and uniqueness of solution to these equations has been addressed by Ayeni (1978), who considered similar set of equations and showed among other results that existence and uniqueness are somewhat well known. In his work, he studied the following system of parabolic equations

$$\left.\begin{aligned} \frac{\partial \phi}{\partial t} &= \Delta\phi + f\left(x,t,\phi,\mathrm{u},\mathrm{v}\right), & x\in R^n,\quad t>0 \\ \frac{\partial u}{\partial t} &= \Delta u + g\left(x,t,\phi,\mathrm{u},\mathrm{v}\right), & x\in R^n,\quad t>0 \\ \frac{\partial v}{\partial t} &= \Delta v + h\left(x,t,\phi,\mathrm{u},\mathrm{v}\right), & x\in R^n,\quad t>0 \end{aligned}\right\} \tag{15}$$

$$\phi\left(x,0\right) = f_0(x)$$
$$\mathrm{u}(x,0) = \mathrm{g}_0(x)$$
$$\mathrm{v}(x,0) = \mathrm{h}_0(x)$$

$$\mathrm{x} = (x_1, x_2, ..., x_n)$$

S1: $f_0(x)$, $g_0(x)$ and $h_0(x)$ are bounded for $x\in R^n$. Each has at most a countable number of discontinuities.

S2: f, g, h satisfies the uniform Lipschitz condition

$$\left|\varphi\left(x,t,\phi_1,\mathrm{u}_1,\mathrm{v}_1\right)-\varphi\left(x,t,\phi_2,\mathrm{u}_2,\mathrm{v}_2\right)\right| \le M\left(\left|\phi_1-\phi_2\right|+\left|u_1-u_2\right|+\left|v_1-v_2\right|\right),\quad (x,t)\in G$$

where

$$G=\left\{\left(x,\mathrm{t}\right):\ x\in R^n,\ 0<t<\tau\right\}$$

Our proof of existence of unique solution of the system of parabolic equations (11) – (13) will be analogous to his proof

Theorem 3.1: There exist a unique solution $u\left(\mathrm{y},\mathrm{t}\right), \theta\left(\mathrm{y},\mathrm{t}\right)$, and $\phi\left(\mathrm{y},\mathrm{t}\right)$ of equations (11) – (13) which satisfy (14)

Lemma 3.1 (Ayeni ((978)**):**

Let $\left(f_0, \mathrm{g}_0, \mathrm{h}_0\right)$ and $\left(f, g, h\right)$ satisfy (**S1**) and (**S2**) respectively. Then there exist a solution of problem (15).

Proof of Lemma 3.1, see Ayeni (1978)

Proof of theorem 3.1: We rewrite equations (11) – (13) as

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial y^2} + f\left(y,t,u,\theta,\phi\right), & y\in R^n,\quad t>0 \\ \frac{\partial \theta}{\partial t} &= \frac{\partial^2 \theta}{\partial y^2} + g\left(y,t,u,\theta,\phi\right), & y\in R^n,\quad t>0 \\ \frac{\partial \phi}{\partial t} &= \frac{\partial^2 \phi}{\partial y^2} + h\left(y,t,u,\theta,\phi\right), & y\in R^n,\quad t>0 \end{aligned}\right\} \tag{16}$$

Where

$$f\left(y,t,u,\theta,\phi\right) = Gr_\phi\phi - \left(M+\frac{1}{K_p}\right)u \tag{17}$$

$$g\left(y,t,u,\theta,\phi\right) = q\theta + Ec\left(\frac{\partial u}{\partial y}\right)^2 \tag{18}$$

$$h\left(y,t,u,\theta,\phi\right) = 0 \tag{19}$$

Ignoring the second term at the right hand side, the fundamental solutions of equations (11) – (13) are (see Toki and Tokis (2007))

$$F(y,\mathrm{t}) = \frac{y}{2t\sqrt{Sc\pi t}}\exp\left(-\frac{y^2}{4Sct}\right) \tag{20}$$

$$G(y,\mathrm{t}) = \frac{\Pr^{\frac{1}{2}} y}{2t\sqrt{\pi t}}\exp\left(-\frac{\Pr y^2}{4t}\right) \tag{21}$$

$$H(y,\mathrm{t}) = \frac{y}{2t\sqrt{\pi t}}\exp\left(-\frac{y^2}{4t}\right) \tag{22}$$

Clearly, $f(y,t,u,\theta,\phi) = Gr_\phi \phi - \left(M + \frac{1}{K_p}\right)u$, $g(y,t,u,\theta,\phi) = q\theta + Ec\left(\frac{\partial u}{\partial y}\right)^2$ and $h(y,t,u,\theta,\phi) = 0$ are Lipschitz continuous. Hence by theorem 3.1, the results follows. This completes the proof.

Properties of Solution

Theorem 3.2: Let $Sc = M = Kp = Gr_\phi = Ec = \Pr = q = 1$ in equation (11)-(13). Then,

$$\frac{\partial u}{\partial t} \geq 0, \quad \frac{\partial \theta}{\partial t} \geq 0, \quad \frac{\partial \phi}{\partial t} \geq 0.$$

In the proof, we shall make use of the following Lemma of Kolodner and Pederson (1966).

Lemma (Kolodner and Pederson (1966)): Let $u(\mathrm{x},\mathrm{t}) = 0(e^{\alpha|x|^2})$ be a solution on $R^n \times [0,t)$ of the differential inequality, $\frac{\partial u}{\partial t} - \Delta u + k(\mathrm{x},\mathrm{t})\mathrm{u} \geq 0$ where k is bounded from below if $u(\mathrm{x},0) \geq 0,$ then $u(\mathrm{x},0) \geq 0$ for all $(x,t) \in R^n \times [0,t_0)$

Proof of Theorem 3.2: Given,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} = \phi - 2u \tag{23}$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} = \theta + \left(\frac{\partial u}{\partial y}\right)^2 \tag{24}$$

$$\frac{\partial \phi}{\partial t} - \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{25}$$

Differentiating with respect to t, we have

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial \phi}{\partial t} - 2\frac{\partial u}{\partial t} \\ &\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{\partial^2 \theta}{\partial y^2}\right) = \frac{\partial \theta}{\partial t} + \frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial y}\right)^2\right) \\ &\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{\partial^2 \phi}{\partial y^2}\right) = 0 \end{aligned}\right\} \tag{26}$$

Let $p = \frac{\partial u}{\partial t}$, $\quad r = \frac{\partial \theta}{\partial t}$ and $w = \frac{\partial \phi}{\partial t}$

Then

$$\frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial y^2} + p \geq 0 \quad \text{since } w \geq 0$$

$$\frac{\partial r}{\partial t} - \frac{\partial^2 r}{\partial y^2} - r \geq 0 \quad \text{since } 2p\frac{\partial p}{\partial y} \geq 0$$

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial y^2} + 0 \times w = 0$$

This can be written as

$$\frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial y^2} + k(y,t)\, p \geq 0$$

$$\frac{\partial r}{\partial t} - \frac{\partial^2 r}{\partial y^2} + k_1(y,t)\, r \geq 0$$

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial y^2} + k_2(y,t)\, w \geq 0$$

Where

$k(\mathrm{y},\mathrm{t}) = 1, \quad k_1(\mathrm{y},\mathrm{t}) = -1, \quad k_2(\mathrm{y},\mathrm{t}) = 0$

Clearly, k and k_1 are bounded from below and k_2 is bounded everywhere. Hence, by Kolodner and Pederson's lemma, $p(y,t) \geq 0$, and $r(y,t) \geq 0$ and $w(y,t) \geq 0$, that is $\frac{\partial u}{\partial t} \geq 0$, $\frac{\partial \theta}{\partial t} \geq 0$ and $\frac{\partial \phi}{\partial t} \geq 0$. This completes the proof.

Theorem 3.3: Let $q > 0$, $Ec = 0$ and $\Pr = 1$ in $\frac{\partial \theta}{\partial t} - \frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2} = Ec\left(\frac{\partial u}{\partial y}\right)^2 + q\theta$. Then $\theta(y,t) \geq 0$ for $(y,t) \in (0,\infty) \times (0,t_0)$, $t_0 > 0$

Proof: Let $q > 0$, $Ec = 0$ and $\Pr = 1$ we have,

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial y^2} = q\theta$$

i.e

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial y^2} - q\theta = 0$$

This can be written as

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial y^2} - K(y,t)\theta = 0$$

Where

$K(y,t) = -q$

Hence, by Kolodner and Perdeson's lemma $\theta(y,t) \geq 0$. This completes the proof.

Conclusion

To examine the properties of solution of the two dimensional flow of a viscous incompressible electrically conducting fluid through a porous medium in an infinite vertical porous plate in the presence of uniform transverse magnetic field (B_0) and constant heat source (Q), we used an approach by Ayeni (1978) and Kolodner and Pederson (1966). Our results revealed that velocity u, mass ϕ and temperature θ are increasing function of time. We can therefore conclude that for cooling of the plate by free convection current $(Gr_\phi > 0)$, velocity, mass and temperature are increasing function of time.

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