

SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH SPECIAL NONLINEARITIES BY ADOMIAN DECOMPOSITION METHOD

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Abstract

In this paper, the generation of Adomian polynomials for certain supposedly difficult nonlinearities and their implementation in standard Adomian decomposition method is reported. The steps involved in the whole process are well elucidated, and the numerical results obtained confirm the accuracy of the method.

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AMS: (2010) Subject classification: 34B05, 34B15, 65L05, 65L10

Introduction

Adomian decomposition method (ADM), which is one of the most reliable semi-analytic methods, was introduced by Adomian (1994). This method prides itself as a viable tool that handles both linear and nonlinear ordinary and partial differential equations. Its application transcends the solution of differential equations, as it has been effectively applied to linear and nonlinear integral as well as integro – differential equations all kinds and types (Wazwaz, 2011). The so-called modified Adomian decomposition method (which is applicable only when the inhomogeneous source term has more than one term) was later reported to simplify matters, especially when there are 'noise' terms in the solution (Adomian, 1994; Wazwaz, 2011).

The only difficulty encounters in the course of using ADM is in the generation of the Adomian polynomials, especially for certain difficult nonlinearities (Hermann and Saravi, 2016). Despite the availability of a robust general formula presented by Adomian (1994) for the generation of the polynomials, researchers still encounter some difficulties in that direction. Duan (2015) presented an extension of ADM to boundary value problems, and most importantly, an algorithm for implementation in Mathematical for the generation of Adomian polynomials. Hermann and Saravi (2016), Yisa and Issa (2018), and a host of other scientists used ADM to solve generalized Emden–Lane–Fowler equation. The peculiarity in the equation is the singularity that exists in its first order term which hinders solving the problem by many known analytical methods. Relativity in performances of variational iteration method (VIM) and ADM were investigated by Yisa (2018), where the two methods were observed to be efficient.

The convergence of any numerical scheme to be used in solving any problem is of central consideration. To that end, Abbaoui and Cherruault (1994) established the general convergence for ADM, while Abdelrazec and Pelinovsky (2011) worked on the convergence of the method basically for IVPs.

In the present work, attention is given to the modalities of generating the Adomian polynomials for some selected nonlinearities of varying degrees of difficulties. The polynomials thus generated are implemented in the solutions of some nonlinear initial value problems (IVPs).

The Adomian Polynomials

Adomian (1994) gave a general formula for generating the so – called Adomian polynomials as

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[N \left(\sum_{j=0}^n \lambda^j y_j \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (2.1)$$

An elegant variation of (2.1) was present in Hermann and Saravi (2016). The derivation of Adomian polynomials through the implementation of (2.1) is presented in section below. It must be noted too, that when $n = 0$, the partial derivative is not implemented, that is $A_0 = N(y_0)$.

Generation of Adomian Polynomial for Certain Nonlinearities

In this section, Adomian polynomials are derived for the types of nonlinearities that are conceived in the present work.

Nonlinearity of the Form $N(y) = y(x)y'(x)$

Consider the nonlinearity

$$N(y) = y(x)y'(x) \quad (3.1)$$

The Adomian polynomials are derived as follows:

$$A_0 = N(y_0) = y_0 y'_0$$

For $n = 1$:

$$A_1 = \frac{1}{1!} \frac{\partial}{\partial \lambda} \left[N \left(\sum_{j=0}^1 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_1 = \frac{\partial}{\partial \lambda} [N(y_0 + \lambda y_1)]_{\lambda=0}$$

This implies

$$A_1 = \frac{\partial}{\partial \lambda} [(y_0 + \lambda y_1)(y'_0 + \lambda y'_1)]_{\lambda=0}$$

Using product rule, we get

$$A_1 = [(y_0 + \lambda y_1)y'_1 + (y'_0 + \lambda y'_1)y_1]_{\lambda=0}$$

$$A_1 = y_0 y'_1 + y'_0 y_1$$

The next Adomian polynomial A_2 is generated as follows:

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} \left[N \left(\sum_{j=0}^2 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_2 = \frac{\partial^2}{\partial \lambda^2} [(y_0 + \lambda y_1 + \lambda^2 y_2)(y'_0 + \lambda y'_1 + \lambda^2 y'_2)]_{\lambda=0}$$

$$A_2 = \frac{1}{2} \frac{\partial}{\partial \lambda} [(y_0 + \lambda y_1 + \lambda^2 y_2)(y'_1 + 2\lambda y'_2) + (y_1 + 2\lambda y_2)(y'_0 + \lambda y'_1 + \lambda^2 y'_2)]_{\lambda=0}$$

$$A_2 = \frac{1}{2} [(y_0 + \lambda y_1 + \lambda^2 y_2)(2y'_2) + (y_1 + 2\lambda y_2)(y'_1 + 2\lambda y'_2) + (y_1 + 2\lambda y_2)(y'_1 + 2\lambda y'_2) + 2y_2(y'_0 + \lambda y'_1 + \lambda^2 y'_2)]_{\lambda=0},$$

which gives

$$A_2 = \frac{1}{2} [2y'_2 y_0 + y_1 y'_1 + y_1 y'_1 + 2y_2 y'_0].$$

Upon simplification, we get

$$A_2 = y'_2 y_0 + y_1 y'_1 + y_2 y'_0.$$

For $n = 3$:

$$A_3 = \frac{1}{3!} \frac{\partial^3}{\partial \lambda^3} \left[N \left(\sum_{j=0}^3 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_3 = \frac{1}{3!} \frac{\partial^3}{\partial \lambda^3} N[(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$A_3 = \frac{1}{6} \frac{\partial^3}{\partial \lambda^3} [(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)(y_0' + \lambda y_1' + \lambda^2 y_2' + \lambda^3 y_3')]_{\lambda=0}$$

Working through carefully and evaluating at $\lambda = 0$, we get

$$A_3 = \frac{1}{6} [6y_3 y_0' + 2y_1 y_2' + 2y_1 y_2' + 2y_2 y_1' + 2y_1 y_2' + 2y_2 y_1' + 2y_2 y_1' + 6y_3 y_0'],$$

which eventually gives

$$A_3 = y_0 y_3' + y_1 y_2' + y_2 y_1' + y_0 y_3'.$$

And so on.

Nonlinearity of the Form $N(y) = y'(x)^2$

Consider the nonlinearity

$$N(y) = (y'(x))^2 = (y')^2 \quad (3.2)$$

Now using (1), we shall generate the corresponding Adomian polynomials as follows:

For $n = 0$:

$$A_0 = N(y_0) = (y_0')^2$$

For $n = 1$:

$$A_1 = \frac{1}{1!} \frac{\partial}{\partial \lambda} \left[N \left(\sum_{j=0}^1 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_1 = \frac{\partial}{\partial \lambda} [N(y_0 + \lambda y_1)]_{\lambda=0}$$

$$A_1 = \left[\frac{\partial}{\partial \lambda} (y_0' + \lambda y_1')^2 \right]_{\lambda=0}$$

$$A_1 = [2(y_0' + \lambda y_1') y_1']_{\lambda=0},$$

which finally gives

$$A_1 = 2y_0' y_1'.$$

For $n = 2$:

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} \left[N \left(\sum_{j=0}^2 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_2 = \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} N[y_0 + \lambda y_1 + \lambda^2 y_2]_{\lambda=0}$$

$$A_2 = \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} [(y_0' + \lambda y_1' + \lambda^2 y_2')^2]_{\lambda=0}$$

$$A_2 = \frac{1}{2} \frac{\partial}{\partial \lambda} [2(y_0' + \lambda y_1' + \lambda^2 y_2')(y_1' + 2\lambda y_2')]_{\lambda=0}$$

$$A_2 = [(y_0' + \lambda y_1' + \lambda^2 y_2')(2y_2') + (y_1' + 2\lambda y_2')^2]_{\lambda=0}$$

And this finally gives

$$A_2 = 2y_2'y_0' + (y_1')^2.$$

For $n = 3$:

$$A_3 = \frac{1}{3!} \frac{\partial^3}{\partial \lambda^3} \left[N \left(\sum_{j=0}^3 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_3 = \frac{1}{3!} \frac{\partial^3}{\partial \lambda^3} N[(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$A_3 = \frac{1}{6} \frac{\partial^3}{\partial \lambda^3} [(y_0' + \lambda y_1' + \lambda^2 y_2' + \lambda^3 y_3')^2]_{\lambda=0}$$

Differentiating and evaluating at $\lambda = 0$, we get

$$A_3 = 2y_0'y_3' + 2y_1'y_2'.$$

Nonlinearity of the Form $N(y) = y''(x)^2$

Consider the nonlinearity

$$N(y) = y''(x)^2 = (y'')^2 \quad (3.3)$$

Using (2.1) here again, the Adomian polynomials corresponding to the nonlinearity are derived as shown below.

For $n = 0$:

$$A_0 = N(y_0) = (y_0'')^2$$

For $n = 1$:

$$A_1 = \frac{1}{1!} \frac{\partial}{\partial \lambda} \left[N \left(\sum_{j=0}^1 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_1 = \frac{\partial}{\partial \lambda} [N(y_0 + \lambda y_1)]_{\lambda=0}$$

$$A_1 = \frac{\partial}{\partial \lambda} [(y_0'' + \lambda y_1'')^2]_{\lambda=0}$$

$$A_1 = [2(y_0'' + \lambda y_1'')y_1'']_{\lambda=0},$$

which finally gives

$$A_1 = 2y_0''y_1''.$$

For subsequent values of n , the polynomials take after those of nonlinearity derived above for

$N(y) = y'(x)^2$ with the minor adjustments in the order of the derivatives, that is changing first order terms to second order terms throughout. So, the polynomials are:

For $n = 2$:

$$A_2 = 2y_2''y_0'' + (y_1'')^2.$$

For $n = 3$:

$$A_3 = 2y_0''y_3'' + 2y_1''y_2''.$$

Nonlinearity of the Form $N(y) = y(x)\ln(y(x))$.

Consider the nonlinearity

$$N(y) = y\ln(y)$$

(3.4)

Using (2.1), the corresponding Adomian polynomials are derived as follows:

For $n = 0$:

$$A_0 = y_0\ln(y_0)$$

For $n = 1$:

$$A_1 = \frac{1}{1!} \frac{\partial}{\partial \lambda} \left[N \left(\sum_{j=0}^1 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_1 = \frac{\partial}{\partial \lambda} [N(y_0 + \lambda y_1)]_{\lambda=0}$$

$$A_1 = \left[\frac{\partial}{\partial \lambda} (y_0 + \lambda y_1) \ln(y_0 + \lambda y_1) \right]_{\lambda=0}$$

$$A_1 = \left[\frac{(y_0 + \lambda y_1)y_1}{(y_0 + \lambda y_1)} + y_1 \ln(y_0 + \lambda y_1) \right]_{\lambda=0}$$

Therefore

$$A_1 = y_1 \ln y_0 + y_1.$$

For $n = 2$:

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} \left[N \left(\sum_{j=0}^2 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_2 = \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} N[y_0 + \lambda y_1 + \lambda^2 y_2]_{\lambda=0}$$

$$A_2 = \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} [(y_0 + \lambda y_1 + \lambda^2 y_2) \ln(y_0 + \lambda y_1 + \lambda^2 y_2)]_{\lambda=0}$$

$$A_2 = \frac{1}{2} [2y_2 + \frac{y_1^2}{2y_0} + 2y_2 \ln y_0]_{\lambda=0}$$

Thus,

$$A_2 = y_2 \ln y_0 + \frac{y_1^2}{2y_0} + y_2.$$

For $n = 3$:

$$A_3 = \frac{1}{3!} \frac{\partial^3}{\partial \lambda^3} \left[N \left(\sum_{j=0}^3 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_3 = \frac{1}{3!} \frac{\partial^3}{\partial \lambda^3} N[(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

$$A_3 = \frac{1}{6} \frac{\partial^3}{\partial \lambda^3} [(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3) \ln(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)]_{\lambda=0}$$

Thus,

$$A_3 = y_3 \ln y_0 + \frac{y_1 y_2}{y_0} - \frac{y_1^3}{6 y_0}.$$

And so on.

Nonlinearity of the Form $N(y) = e^{\frac{y(x)}{2}}$

Consider the nonlinearity

$$N(y) = e^{\frac{y(x)}{2}} \quad (3.5)$$

The Adomian decomposition corresponding to the nonlinearity are derived using (1) as follows:

For $n = 0$:

$$A_0 = N(y_0) = e^{\frac{y_0}{2}}.$$

For $n = 1$:

$$A_1 = \frac{1}{1!} \frac{\partial}{\partial \lambda} \left[N \left(\sum_{j=0}^1 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_1 = \frac{\partial}{\partial \lambda} [N(y_0 + \lambda y_1)]_{\lambda=0}$$

This implies

$$A_1 = \frac{\partial}{\partial \lambda} [e^{\frac{y_0 + \lambda y_1}{2}}]_{\lambda=0}$$

$$A_1 = y_1 [e^{\frac{y_0 + \lambda y_1}{2}}]_{\lambda=0}$$

Therefore,

$$A_1 = y_1 e^{\frac{y_0}{2}}.$$

For $n = 2$:

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} \left[N \left(\sum_{j=0}^2 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_2 = \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} N[y_0 + \lambda y_1 + \lambda^2 y_2]_{\lambda=0}$$

$$A_2 = \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} [e^{\frac{y_0 + \lambda y_1 + \lambda^2 y_2}{2}}]_{\lambda=0}$$

$$A_2 = [\frac{1}{2} y_2 e^{\frac{y_0 + \lambda y_1 + \lambda^2 y_2}{2}} + \frac{1}{8} (y_1 + 2\lambda y_2)^2 e^{\frac{y_0 + \lambda y_1 + \lambda^2 y_2}{2}}]_{\lambda=0}$$

This gives

$$A_2 = \frac{1}{2}y_2 e^{\frac{y_0}{2}} + \frac{1}{2}y_2 e^{\frac{y_0}{2}} y_1^2.$$

For $n = 3$:

$$A_3 = \frac{1}{3!} \frac{\partial^3}{\partial \lambda^3} \left[N \left(\sum_{j=0}^3 \lambda^j y_j \right) \right]_{\lambda=0}$$

$$A_3 = \frac{1}{6} \frac{\partial^3}{\partial \lambda^3} \left[e^{\frac{y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3}{2}} \right]_{\lambda=0}$$

$$A_3 = \frac{1}{2}y_3 e^{\frac{y_0}{2}} + \frac{1}{4}y_1 y_3 e^{\frac{y_0}{2}} + \frac{1}{48}y_1^3 e^{\frac{y_0}{2}}.$$

A Review of Adomian Decomposition Method

A brief review of ADM is presented in this section for the sake of completeness, interested reader can find the details in Adomian (1994), Wazwaz (2011), Hermann and Saravi (2016), just to mention a few.

Consider the n th order nonlinear initial value problem

$$Ly(x) + N(y(x)) + Ry(x) = g(x) \quad (4.1a)$$

With the initial conditions

$$y(0) = \alpha_1, y'(0) = \alpha_2, \dots, y^{(n-1)}(0) = \alpha_n, \quad (4.1b)$$

where L is an n th order linear differential operator, N is the nonlinear operator, R is the remaining linear term and $g(x)$ is the inhomogeneous source term.

The initial approximation $y_0(x)$ is obtained as follows:

$$y_0(x) = \psi_0(x) + \beta(x) \quad (4.2)$$

The $\psi_0(x)$ in (8) is derived from the Taylor's series using the initial conditions as follows:

$$\psi_0(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \dots + \frac{x^{n-1}}{(n-1)!}y^{(n-1)}(0) \quad (4.3)$$

That is

$$\psi_0(x) = \alpha_1 + \alpha_2 x + \frac{\alpha_3 x^2}{2!} + \dots + \frac{\alpha_n x^{n-1}}{(n-1)!} \quad (4.4)$$

On the other hand, $\beta(x)$ is the result of application of L^{-1} (which is the inverse of the linear differential operator L) to $g(x)$ in (4.1a). Thus

$$\beta(x) = L^{-1}g(x) \quad (4.5)$$

The initial approximation is therefore obtained through adding (4.3) to (4.4), so we get

$$y_0(x) = \alpha_1 + \alpha_2 x + \frac{\alpha_3 x^2}{2!} + \dots + \frac{\alpha_n x^{n-1}}{(n-1)!} + L^{-1}g(x) \quad (4.6)$$

The final solution is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \dots \quad (4.7)$$

The other members $y_1(x)$, $y_2(x)$, ... are obtained from the recurrence relation

$$y_{k+1}(x) = -L^{-1}(Ry) - L^{-1}(N(y)), \quad k = 0, 1, 2, \dots \quad (4.8)$$

That is

$$y_{k+1}(x) = - \int_0^x \int_0^{x_{n-1}} \dots \int_0^{x_2} \int_0^{x_1} Ry_k(t) dt dx_1 \dots dx_{n-1} - \int_0^x \int_0^{x_{n-1}} \dots \int_0^{x_2} \int_0^{x_1} A_k(t) dt dx_1 \dots dx_{n-1} \quad (4.9)$$

where $A_k(x)$ with $k = 1, 2, 3, \dots$ are the Adomian polynomials corresponding to the given nonlinearity.

Numerical Experiments

In this section, few problems are considered for the implementation of the algorithm explained in the preceding sections.

Problem 1 (Hermann and Saravi (2016))

Consider the second order nonlinear inhomogeneous IVP

$$y''(x) + 2y(x)y'(x) - y(x) = \sinh(2x), \quad y(0) = 0, \quad y'(0) = 1.$$

Solution

$$\begin{aligned} y_0(x) &= \psi_0(x) + \beta(x) \\ \psi_0(x) &= y(0) + xy'(0) = 0 + x \cdot 1 = x \\ \beta(x) &= \int_0^x \int_0^\tau \sinh(2t) dt d\tau \\ \beta(x) &= -\frac{1}{2}x + \frac{1}{4}\sinh(2x) \end{aligned}$$

Therefore,

$$y_0(x) = x - \frac{1}{2}x + \frac{1}{4}\sinh(2x)$$

Thus

$$y_0(x) = \frac{x}{2} + \frac{1}{4}\sinh(2x).$$

The recurrence relation is given by

$$y_{k+1}(x) = -2 \int_0^x \int_0^\tau A_k(t) dt d\tau + \int_0^x \int_0^\tau y_k(t) dt d\tau, \quad k = 0, 1, 2, \dots$$

where $A_k(x)$ are the Adomian polynomials derived for the nonlinearity (2).

$$\begin{aligned} y_1(x) &= -2 \int_0^x \int_0^\tau A_0(t) dt d\tau + \int_0^x \int_0^\tau y_0(t) dt d\tau \\ y_1(x) &= -2 \int_0^x \int_0^\tau y_0(t) y_0'(t) dt d\tau + \int_0^x \int_0^\tau y_0(t) dt d\tau \end{aligned}$$

$$\begin{aligned} y_1(x) &= \\ &-2 \int_0^x \int_0^\tau \left(\frac{t}{2} + \frac{1}{4}\sinh(2t) \right) \left(\frac{1}{2} + \frac{1}{2}\cosh(2t) \right) dt d\tau + \int_0^x \int_0^\tau \left(\frac{t}{2} + \frac{1}{4}\sinh(2t) \right) dt d\tau \end{aligned}$$

After simplification, we have

$$y_1(x) = -\frac{3x}{32} - \frac{1}{8}x\cosh(2x) + \frac{1}{8}\cosh(x)\sinh(x) + \frac{1}{16}\sinh(2x) - \frac{1}{128}\sinh(4x).$$

Also,

$$y_2(x) = -2 \int_0^x \int_0^\tau A_1(t) dt d\tau + \int_0^x \int_0^\tau y_1(t) dt d\tau$$

Using the appropriate Adomian polynomials as we have it in (2), we get

$$y_1(x) = -2 \int_0^x \int_0^\tau (y_0'(t)y_1(t) + y_0(t)y_1'(t)) dt d\tau + \int_0^x \int_0^\tau y_1(t) dt d\tau$$

Making necessary substitutions and simplifying gives

$$y_2(x) = -\frac{31x}{512} + \frac{x^3}{64} - \frac{17}{128}x\cosh(2x) + \frac{5}{512}x\cosh(4x) + \frac{115\sinh(2x)}{1024} + \frac{1}{16}x^2\sinh(2x) - \frac{11\sinh(4x)}{1024} + \frac{\sinh(6x)}{3072}.$$

The solution is given by

$$y(x) = \sum_{n=1}^{\infty} y_n(x)$$

Thus,

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \dots$$

Therefore,

$$y(x) = \frac{177x}{512} + \frac{x^3}{64} - \frac{33}{128}x\cosh(2x) + \frac{5}{512}x\cosh(4x) + \frac{1}{8}\cosh(x)\sinh(x) + \frac{435\sinh(2x)}{1024} + \frac{1}{16}x^2\sinh(2x) - \frac{19\sinh(4x)}{1024} + \frac{\sinh(6x)}{3072}.$$

Problem 2 (Hermann & Saravi, 2016)

Consider the third order nonlinear inhomogeneous IVP

$$y'''(x) + y''(x)^2 + y'(x)^2 = 1 - \cos(x)$$

with the associated boundary conditions

$$y(0) = y''(0) = 0, \quad y'(0) = 1.$$

Solution

Here, the initial approximation $y_0(x)$ is obtained using

$$y_0(x) = \psi_0(x) + \beta(x)$$

where

$$\psi_0(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) = x,$$

and

$$\beta(x) = \int_0^x \int_0^\tau \int_0^t (1 - \cos(\xi)) d\xi dt d\tau$$

which gives

$$\beta(x) = -x + \frac{x^3}{6} + \sin(x).$$

Thus,

$$y_0(x) = x - x + \frac{x^3}{6} + \sin(x)$$

Therefore,

$$y_0(x) = \frac{x^3}{6} + \sin(x).$$

The subsequent members of the series are obtained via the recurrence relation

$$y_{k+1}(x) = -\int_0^x \int_0^\tau \int_0^t A_k(\xi) d\xi dt d\tau - \int_0^x \int_0^\tau \int_0^t B_k(\xi) d\xi dt d\tau, k = 0, 1, 2, \dots$$

where $A_k(x)$ and $B_k(x)$ are the Adomian polynomials corresponding to the nonlinearities $N(y) = (y'')^2$ and $N(y) = (y')^2$ respectively.

Hence,

$$y_1(x) = -\int_0^x \int_0^\tau \int_0^t A_0(\xi) d\xi dt d\tau - \int_0^x \int_0^\tau \int_0^t B_0(\xi) d\xi dt d\tau$$

Using the corresponding Adomian polynomials, we have

$$y_1(x) = -\int_0^x \int_0^\tau \int_0^t (y_0''(\xi))^2 d\xi dt d\tau - \int_0^x \int_0^\tau \int_0^t (y_0'(\xi))^2 d\xi dt d\tau$$

This eventually gives

$$y_1(x) = 10x - \frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{840} + 8x\cos(x) - 18\sin(x) + x^2\sin(x).$$

Also, we have

$$y_2(x) = -\int_0^x \int_0^\tau \int_0^t A_1(\xi) d\xi dt d\tau - \int_0^x \int_0^\tau \int_0^t B_1(\xi) d\xi dt d\tau$$

Using the appropriate Adomian polynomials, we have

$$y_1(x) = -\int_0^x \int_0^\tau \int_0^t 2y_0''(\xi)y_1''(\xi) d\xi dt d\tau - \int_0^x \int_0^\tau \int_0^t 2y_0'(\xi)y_1'(\xi) d\xi dt d\tau$$

This gives

$$\begin{aligned} y_2(x) = & -584x + \frac{7}{3}x^3 - \frac{x^5}{6} + \frac{x^7}{180} + \frac{11x^9}{30240} + \frac{x^{11}}{118800} - 1020x\cos(x) + \frac{148}{3}x^3\cos(x) \\ & - \frac{2}{5}x^5\cos(x) + \frac{1}{4}x\cos(2x) + 1606\sin(x) - 295x^2\sin(x) + \frac{16}{3}x^4\sin(x) \\ & - \frac{1}{60}x^6\sin(x) - \frac{9}{4}\cos(x)\sin(x). \end{aligned}$$

The final result is obtained using

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

Thus,

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \dots$$

Therefore,

$$\begin{aligned} y(x) = & -574x + \frac{7}{3}x^3 - \frac{11x^5}{60} + \frac{11x^7}{2520} + \frac{11x^9}{30240} + \frac{x^{11}}{118800} - 1020x\cos(x) + \frac{148}{3}x^3\cos(x) \\ & - \frac{2}{5}x^5\cos(x) + \frac{1}{4}x\cos(2x) + 1589\sin(x) - 294\sin(x) + \frac{16}{3}x^4\sin(x) \\ & - \frac{1}{60}x^6\sin(x) - \frac{9}{4}\cos(x)\sin(x). \end{aligned}$$

Discussion of Results and Conclusion

The derivation of Adomian polynomials for some strong nonlinearities has been presented. The techniques involved are well elucidated in a manner that facilitates quick understanding by anyone that has the need to derive the polynomials for whatever form of nonlinearity. The numerical examples presented fall in the category of the said nonlinearities, just to be able to demonstrate its usage in the Adomian decomposition method. The correctness of the solutions of the numerical problems is easily verified by implementing the associated initial conditions.

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