MODIFIED FAMILY OF GENERALIZED MULTIVARIATE SEMI PARETO (III) DISTRIBUTIONS AND ITS CHARACTERIZATION PROPERTY

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Abstract
This paper gives Marshall-Olkin form of generalized multivariate semi-Pareto (III) which is already studied by Yeh (2007). Method of finite sample minima is used to characterize this new family of distribution. This characterization property is given in the form of theorem. The theorem is later proved through general and particular solutions of Euler's functional equation for homogeneous functions. Mat lab programming language is use to simulate the data which is then fits into this family of distributions. The result obtained is later compared with the one derived from the distribution given in Yeh (2007).

Keywords: Generalized multivariate semi-Pareto distribution; Marshall-Olkin form of distribution; Joint survival function; Characterization property; Euler's functional equation; Finite sample minima

Introduction
Generalized multivariate semi-Pareto distribution denoted by $GMSP^{(k)}$, like any other Pareto distribution plays important role in daily socio-economic activities. It also fits well to the upper tails of some multivariate continuous income data as well as some other socio-economic multivariate variables.

The univariate semi-Pareto distribution was first introduced by Pillai (1991). Some special bivariate semi-Pareto distribution with homogeneous scale parameters were studied by Balakrishna and Jayakumar (1997). Yeh (2004a) had studied some characterization properties of generalized multivariate Pareto (III) distribution, $MP^{(n)}$ (III) already discussed by Arnold (1983), through the geometric minimization procedures. In the same year, Yeh (2004b) developed generalized Marshall-Olkin type of multivariate Pareto distribution. All these results were extended to the generalized multivariate semi-Pareto (GMSP $^{(n)}$) distribution by Yeh (2007). Characterizations of generalized multivariate semi-Pareto distribution, using the method of finite sample minima, were also studied by Yeh (2007).

The proofs of these characteristics given in Yeh 2007 were based on the general and the particular solutions of the Euler's functional equations of n$\geq$1 variables.

Thomas and Jose (2002) introduced and studied the univariate Marshall-Olkin Pareto processes. After that, Thomas and Jose (2004) developed a new family of distributions that were earlier studied by Marshall-Olkin (1997) which is similar to those of Pillai, et al (1995). In their research work, they developed the Marshall-Olkin bivariate semi-Pareto distribution as a generalization of the bivariate semi-Pareto distribution of Balakrishna and Jayakumar (1997). Miroslav, *et al* (2008) presented and studied a bivariate minification process with Marshall-Olkin exponential distribution. Gulumbe, *et al* (2014) developed Marshall-Olkin form of generalized multivariate Pareto (III) with univariate Pareto (III) marginal and multivariate Pareto (III) distributions. Characterizations Properties of these distributions were given in the form of theorems the prove of which were also presented. Recently, Umar and Gulumbe (2018) give the extension of existing families of generalized multivariate Pareto (III)

Distributions earlier presented in Yeh (2007) and Gulumbe *et al* (2014). In their paper, the parameter added is allowed to take on infinite integral values which indeed provide room to accommodate more values, thus capable of tackling more complex problems that may arise.

In this paper, we develop Marshall-Olkin generalized multivariate semi-Pareto distribution, denoted by $MO-GMSP^{(k)}$, as an extension of generalized multivariate semi-Pareto distribution of Yeh (2007) as well as Gulumbe, *et al* (2014). Characterization property of this family of distributions is deduced using the method of finite sample minima and proved through general and particular solutions of Euler's functional equations of homogeneous functions. We later used MATLAB lab programming language to simulate the data which is then fit into this family of distributions. The result obtained is also compared with the one derived from the distribution given in the literature.

Marshall-Olkin Generalized Multivariate Semi-Pareto Distribution

According to Yeh (2007), a broad class of multivariate semi-Pareto distributions is generalized multivariate semi-Pareto distribution, this class of distributions is denoted by $GMSP^{(k)}\left(\underline{\alpha}, \underline{p}\right)$. He defined it as follows:

Definition 2.1.1

A random vector $\underline{X} = \left(X_1, X_2, ..., X_k\right)$ is said to follow a generalized multivariate semi-Pareto distribution if each X_i in $\underline{X}$ is univariate semi-Pareto distributed denoted as $SP\left(\alpha_i, \underline{p} = \left(p_1, p_2, ..., p_l\right)\right)$ distribution. That is, the survival function of each x_i is of the form

$$\overline{G}_i\left(x_i\right) = \frac{1}{1+\varphi_i\left(x_i\right)} \qquad \text{... (2.1.1)}$$

where $\varphi_i\left(x_i\right)$ satisfies the functional equation

$$\varphi_i\left(x_i\right) = \sum_{j=1}^{l} p_j \varphi_i\left(p_j^{-1/\alpha_i} x_i\right) \qquad \text{... (2.1.2)}$$

for any $x_i > 0$, and some $\alpha_i > 0,\ i = 1, 2, ..., k,$ and $0 < p_j < 1, \quad j = 1, 2, ..., l,$ and $l \in N$ is a positive integer, $l \geq 2$. Such a random vector $\underline{X}$ is denoted by $\underline{X} \sim GMSP^{(k)}\left(\underline{\alpha}, \underline{p}\right)$, where $\underline{\alpha} = \left(\alpha_1, \alpha_2, ..., \alpha_k\right)$ and $\underline{p} = \left(p_1, p_2, ..., p_l\right)$.

The following definition is an extension of Thomas and Jose (2004) work that gave the Marshall-Olkin bivariate distribution. In this paper, we give the definition in the case of Marshall-Olkin generalized multivariate distribution.

Let $\underline{X} = \left(X_1, X_2, ..., X_k\right)$ be a k-variate random vector with joint survival function as given in equations (2.1.1) and (2.1.2). Then the Marshall-Olkin generalized multivariate distribution is of the form

$$\overline{F}_{\underline{X}}\left(\underline{X}\right) = \frac{\beta \overline{G}_{\underline{X}}\left(\underline{X}\right)}{1-\left(1-\beta\right)\overline{G}_{\underline{X}}\left(\underline{X}\right)}, \qquad \underline{X}\left(x_1, x_2, ... x_k\right),\ x_i \geq 0,\ i = 1, 2, ..., k,\ \ 0 < \beta < 1,$$

…(2.1.3)

This family of distributions is called Marshall-Olkin generalized multivariate family of distributions.

Equation (2.1.1) above, can be seen in Umar (2011).

Definition 2.1.2
From equations (2.1.1), (2.1.2) and (2.1.3), Marshall-Olkin generalized multivariate semi-Pareto distribution which is denoted by $MO-GMSP^{(k)}\left(\underline{\alpha}, \underline{p}\right)$, has the survival function given as follows:

$$\overline{F}_i\left(x_i\right)=\frac{\beta\overline{G}_i\left(x_i\right)}{1-\left(1-\beta\right)\overline{G}_i\left(x_i\right)}, \qquad x_i \geq 0, \qquad 0<\beta<1.$$

$$=\frac{\beta\left\{1+\varphi_i\left(x_i\right)\right\}^{-1}}{1-\left(1-\beta\right)\left\{1+\varphi_i\left(x_i\right)\right\}^{-1}}, \qquad x_i \geq 0; \qquad 0<\beta<1$$

$$=\frac{\beta}{\beta+\varphi_i\left(x_i\right)}, \qquad \underline{X}>0; \qquad 0<\beta<1.$$

Hence,

$$\overline{F}_i\left(x_i\right)=\frac{1}{1+\frac{1}{\beta}\varphi_i\left(x_i\right)}, \qquad x_i>0; \qquad 0<\beta<1. \qquad \ldots(2.1.4)$$

Satisfying

$$\varphi_i\left(x_i\right)=\sum_{j=1}^{l} p_j\varphi_i\left(p_j^{-1/\alpha_i}x_i\right), \ldots \text{ (2.1.5)}$$

for any $x_i>0$, and some $\alpha_i>0,\ i=1,2,\ldots,k,$ and $0<p_j<1,\quad j=1,2,\ldots,l,$ and $l\in N$ is a positive integer with $l\geq 2$.

Equations (2.1.4) and (2.1.5) above, give the Marshall-Olkin generalized multivariate semi Pareto distributions.

Characterization Property
In this section, we give the theorem that provides us with the characterization property of Marshall-Olkin generalized multivariate semi-Pareto distribution using the method of finite sample minima. The proof of this theorem will also be seen in this section.

Theorem 3.1.1: Let $\left\{\underline{X}^j=\left(X_1^j, X_2^j,\ldots,X_k^j\right)\right\}_{j=1}^{l}$, $l\left(\geq 2\right)$ be k-variate independent random vectors with support on $\left(0,\infty\right)^k$ and common joint survival function, $\overline{F}_{\underline{X}}\left(\underline{X}\right)$. Let $b_j,\quad j=1,2,\ldots,l,$ be l constants such that $b_j>0$. Also for each j, let the k-variate random vectors be defined as $\underline{Y}^j=\min\limits_{1\leq h\leq l,\ h\neq j}\left\{b_h^{-1}\underline{X}^h\right\}\equiv\left(Y_1^j,Y_2^j,\ldots,Y_k^j\right)$. The i^{th} coordinate of $\underline{Y}^j$ that is Y_i^j, $i=1,2,\ldots,k,$ is also defined as

$$Y_i^j=\min\left\{b_1^{-1}X_i^1, b_2^{-1}X_i^2,\ldots,b_{j-1}^{-1}X_i^{j-1}, b_{j+1}^{-1}X_i^{j+1},\ldots b_l^{-1}X_i^l\right\}. \qquad \ldots(3.1.1)$$

Assume that $a_j,\quad j=1,2,\ldots,l,$ satisfy $\sum\limits_{j=1}^{l}a_j=1$. Then the following two statements are equivalent:

(1) The i^{th} marginal survival function of $\overline{F}_{\underline{X}}\left(\underline{X}\right)$ satisfies the functional equation

$$\overline{F}_i\left(x_i\right)=\frac{\Pi_{j=1}^{l}\overline{F}_i\left(b_jx_i\right)}{\sum\limits_{j=1}^{l}a_j\overline{F}_{Y_i^j}\left(x_i\right)} \qquad \ldots(3.1.2)$$

(2) The common joint survival function of $\underline{X}$, $\overline{F}_{\underline{X}}\left(\underline{X}\right)$ is Marshall-Olkin generalized multivariate semi-Pareto $\left(MO-GMSP^{(k)}\left(\underline{\alpha},\underline{p}\right)\right)$ distribution as given in definition (2.1.1).

Proof: To prove theorem 3.1.1 above, we proceed as follows:
Let us assume statement (1) is true and show that statement (2) is also true.

Suppose $\overline{F}_i\left(x_i\right)=\dfrac{\Pi_{j=1}^{l}\overline{F}_i\left(b_jx_i\right)}{\sum_{j=1}^{l}a_j\overline{F}_{Y_i^j}\left(x_i\right)}$, where

$$\overline{F}_{Y_i^j}\left(x_i\right)=P\left(\min_{1\le h\le l,\ h\ne j}b_h^{-1}X_i^h>x_i\right)=\prod_{1\le h\le l,\ h\ne j}\overline{F}_i\left(b_hx_i\right) \qquad \ldots(3.1.3)$$

Therefore,

$$\overline{F}_i\left(x_i\right)=\frac{1}{\sum_{j=1}^{\ell}a_j\frac{1}{\overline{F}_i\left(b_jx_i\right)}} \qquad \ldots(3.1.4)$$

Let $\varphi_i\left(x_i\right)=\dfrac{\beta\left(1-\overline{F}_i\left(x_i\right)\right)}{\overline{F}_i\left(x_i\right)}$. So, $\overline{F}_i\left(x_i\right)=\dfrac{1}{1+\frac{1}{\beta}\varphi_i\left(x_i\right)}$, then equation (3.1.4) becomes

$$\overline{F}_i\left(x_i\right)=\frac{1}{1+\frac{1}{\beta}\varphi_i\left(x_i\right)}=\frac{1}{\sum_{j=1}^{l}a_j\left(1+\frac{1}{\beta}\varphi_i\left(b_jx_i\right)\right)}=\frac{1}{\sum_{j=1}^{l}a_j+\frac{1}{\beta}\sum_{j=1}^{l}a_j\varphi_i\left(b_jx_i\right)}=\frac{1}{1+\frac{1}{\beta}\sum_{j=1}^{l}a_j\varphi_i\left(b_jx_i\right)}.$$

Therefore, $\varphi_i\left(x_i\right)=\sum_{j=1}^{l}a_j\varphi_i\left(b_jx_i\right)$.

If $a_j=p_j$ and $b_j=p_j^{-1/\alpha_i}$, then $\varphi_i\left(x_i\right)=\sum_{j=1}^{l}p_j\varphi_i\left(p_j^{-1/\alpha_i}x_i\right)$. Hence, the functional equation given in definition 2.1.3 is satisfied. Hence random vector $\underline{X}$ is distributed as Marshall-Olkin generalized multivariate semi-Pareto as stated in definition 2.1.3.
Hence, statement (2) followed.
On the other hand, let us assume statement (2) is true and show that statement (1) hold.
Suppose that the common joint survival function of $\underline{X}$, $\overline{F}_{\underline{X}}\left(\underline{X}\right)$ is Marshall-Olkin generalized multivariate semi-Pareto distribution $\left(MO-GMSP\right)$ $\left[\alpha_i,\quad \underline{p}=\left(p_1,p_2,\ldots,p_l\right)\right]$. That is, for each i^{th} marginal function of $\overline{F}_{\underline{X}}\left(\underline{X}\right)$, $\overline{F}_i\left(x_i\right)=\dfrac{1}{1+\frac{1}{\beta}\varphi_i\left(x_i\right)}$ and functional equation $\varphi_i\left(x_i\right)=\sum_{j=1}^{l}p_j\varphi_i\left(p_j^{-1/\alpha_i}x_i\right)$ is satisfied. Then, if we choose $a_j=p_j$ and $b_j=p_j^{-1/\alpha_i}$ with $\sum_{j=1}^{l}a_j=1$, we have

$\overline{F}_i\left(x_i\right)=\dfrac{1}{1+\frac{1}{\beta}\varphi_i\left(x_i\right)}=\dfrac{1}{1+\frac{1}{\beta}\sum_{j=1}^{l}a_j\varphi_i\left(b_jx_i\right)}$ and therefore

$$\overline{F}_i\left(x_i\right)=\frac{1}{1+\frac{1}{\beta}\varphi_i\left(x_i\right)}=\frac{1}{\sum_{j=1}^{l}a_j+\frac{1}{\beta}\sum_{j=1}^{l}a_j\varphi_i\left(b_jx_i\right)}=\frac{1}{\sum_{j=1}^{l}a_j\left(1+\frac{1}{\beta}\varphi_i\left(b_jx_i\right)\right)}=\frac{1}{\sum_{j=1}^{l}a_j\frac{1}{\overline{F}_i\left(b_jx_i\right)}}.$$

Hence, $\overline{F}_i\left(b_jx_i\right)=\overline{F}_i\left(x_i\right)\sum_{j=1}^{l}a_j$. Multiplying both sides by $\prod_{1\le h\le l,\, h\ne j}\overline{F}_i\left(b_hx_i\right)$ yields

$$\prod_{j=1}^{l}\overline{F}_i\left(b_jx_i\right)=\overline{F}_i\left(x_i\right)\sum_{j=1}^{l}a_j\prod_{1\le h\le l,\, h\ne j}\overline{F}_i\left(b_hx_i\right).$$

…(3.1.5)

Therefore, from equation (3.1.5) and the fact that

$\overline{F}_{Y_i^j}\left(x_i\right)=P\left(\min_{1\le h\le l,\, h\ne j}b_h^{-1}X_i^h>x_i\right)=\prod_{1\le h\le l,\, h\ne j}\overline{F}_i\left(b_hx_i\right)$, we have $\overline{F}_i\left(x_i\right)=\dfrac{\prod_{j=1}^{l}\overline{F}_i\left(b_jx_i\right)}{\sum_{j=1}^{l}a_j\overline{F}_{Y_i^j}\left(x_i\right)}$.

Consequently, statement (1) is true.

Hence, the theorem that gives the characterization property of Marshall-Olkin generalized multivariate semi-Pareto distribution is completely proved.

Fitting Data into New Family of Distributions

In this section, MATLAB programming language is employed to simulate the data as well as use it to fit that data into this new family of distributions by using particular solution of Euler's functional equation of homogeneous function.

According to Euler (1925), Castillo and Ruiz-Cobo (1992) as well as Yeh (2007), the particular solution for the generalized Euler's functional equation is $\varphi_i\left(x_i\right)=\left(\frac{x_i}{\sigma_i}\right)^{\alpha}$ for some $\alpha>0$, and $\sigma_i>0$. Therefore, as an extension to Marshall-Olkin form, we have $\varphi_i\left(x_i\right)=\frac{1}{\beta}\left(\frac{x_i}{\sigma_i}\right)^{\alpha}$ for some $\alpha>0,\ \sigma_i>0$ and $0<\beta<1$.

Table 4.1 shows the various values of $\overline{F}_i\left(x_i\right)=\left\{1+\frac{1}{\beta}\left(\frac{x_i}{\sigma_i}\right)^{\frac{1}{\gamma_i}}\right\}^{-1}$, that is, Marshall-Olkin generalized multivariate Pareto (III) with univariate Pareto (III) marginal distributions, for different values of $x_i>0,\sigma_i>0,\gamma_i>0$ as well as for $\beta=0.02,\ \ \beta=0.95$ and $\beta=1$. The first two values of $\beta^{`s}$ are two almost extreme ends values of Marshall-Olkin generalized multivariate Pareto (III) with univariate Pareto (III) marginal distributions, while last value of $\beta\left(\beta=1\right)$, is for generalized multivariate Pareto (III) with univariate Pareto (III) marginal distributions of Yeh (2007).

Table 4.1: A table showing the various values of Marshall-Olkin generalized multivariate Pareto (III) with univariate Pareto (III) marginal distributions for $\beta=0.02,\beta=0.95\ and\ \beta=1$.

				$\overline{F}_i\left(x_i\right)$		
i	x_i	σ_i	γ_i	**β = 0.02**	**β = 0.95**	**β = 1**
1	0.1000	0.1000	0.1000	0.0004	0.0190	0.0200
2	10.0000	0.1633	0.1467	0.0003	0.0152	0.0160
3	20.0000	0.2267	0.1933	0.0003	0.0121	0.0128

4	30.0000	0.2900	0.2400	0.0002	0.0097	0.0102
5	40.0000	0.3533	0.2867	0.0002	0.0077	0.0081
6	50.0000	0.4167	0.3333	0.0001	0.0062	0.0065
7	60.0000	0.4800	0.3800	0.0001	0.0049	0.0052
8	70.0000	0.5433	0.4267	0.0001	0.0039	0.0041
9	80.0000	0.6067	0.4733	0.0001	0.0031	0.0033
10	90.0000	0.6700	0.5200	0.0001	0.0025	0.0026
11	100.0000	0.7333	0.5667	0.0000	0.0020	0.0021
12	110.0000	0.7967	0.6133	0.0000	0.0016	0.0016
13	120.0000	0.8600	0.6600	0.0000	0.0012	0.0013
14	130.0000	0.9233	0.7067	0.0000	0.0010	0.0010
15	140.0000	0.9867	0.7533	0.0000	0.0008	0.0008
16	150.0000	1.0500	0.8000	0.0000	0.0006	0.0007
17	160.0000	1.1133	0.8467	0.0000	0.0005	0.0005
18	170.0000	1.1767	0.8933	0.0000	0.0004	0.0004
19	180.0000	1.2400	0.9400	0.0000	0.0003	0.0003
20	190.0000	1.3033	0.9867	0.0000	0.0002	0.0003
21	200.0000	1.3667	1.0333	0.0000	0.0002	0.0002
22	210.0000	1.4300	1.0800	0.0000	0.0002	0.0002
23	220.0000	1.4933	1.1267	0.0000	0.0001	0.0001
24	230.0000	1.5567	1.1733	0.0000	0.0001	0.0001
25	240.0000	1.6200	1.2200	0.0000	0.0001	0.0001
26	250.0000	1.6833	1.2667	0.0000	0.0001	0.0001
27	260.0000	1.7467	1.3133	0.0000	0.0001	0.0001
28	270.0000	1.8100	1.3600	0.0000	0.0000	0.0000
29	280.0000	1.8733	1.4067	0.0000	0.0000	0.0000
30	290.0000	1.9367	1.4533	0.0000	0.0000	0.0000
31	300.0000	2.0000	1.5000	0.0000	0.0000	0.0000

The graph showing the above information for various values of β , can be seen in figure 4.1.

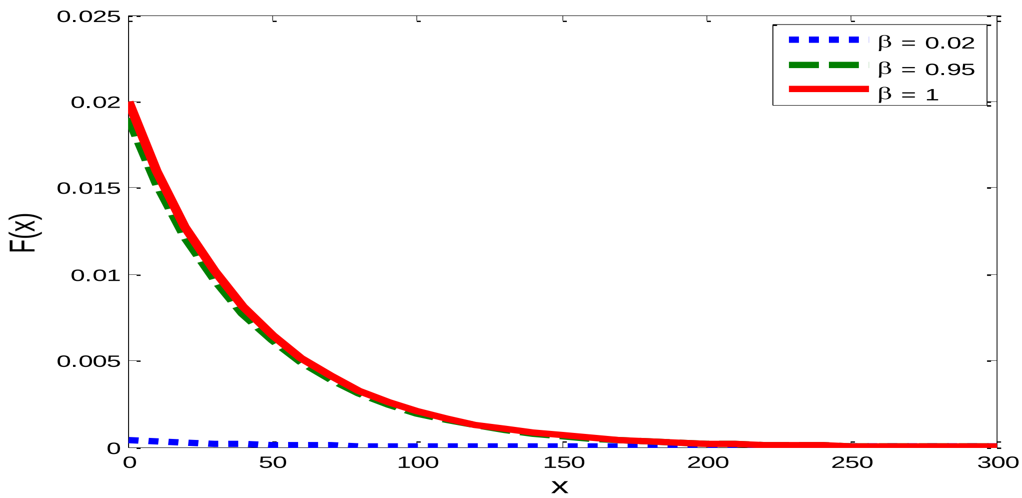

Figure 4.1: A graph showing the various values of Marshall-Olkin generalized multivariate Pareto (III) with univariate Pareto (III) marginal distributions for $\beta = 0.02, \beta = 0.95$ *and* $\beta = 1$.

Discussion of the Result

Pareto distributions are generally used to describe the allocations of wealth among individuals. Since it seemed to show rather well the way that larger portion of the wealth of any society is owned by a smaller percentage of the people in that society. This idea is expressed more simply as the Pareto principle or 80-20 rule which says that 20% of the people controls 80% of the wealth.

The Marshall-Olkin form of these distributions derived in this research work will make analysis of such data of real life situations easier. As new parameter is introduced, and assumes infinitely many values more possible situations of real life can be handle by this new family of distributions.

The Characterization property of this new family of distributions provides us with validation of these distributions. This is so since characterization property of the distributions given in Thomas and Jose (2002, 2004) and Yeh (2007) also hold in the new family of distributions derived in this paper.

Considering Table 4.1 and its corresponding graph given in figure (4.1), it can be seen that the probability or fraction of the population that owns a small amount of wealth per person is higher when parameter $\beta = 1,$ that is in the distributions given in the literature. In the new family of distributions obtained in this paper, that is Marshall-Olkin form of the distributions given in Yeh (2007), in which parameter β is within the interval $0 < \beta < 1,$ it is observed that the probability or fraction of population that owns a small amount of wealth per person is decreasing with decreases in the newly introduced parameter, $\beta.$ Indeed, this fraction or percentage of population is approaching zero exponentially as the value of parameter, β approaches zero.

Similarly from the graph, it can also be seen that the probability or fraction of the population that owns a small amount of wealth per person is high when the entire wealth is small, and then decreases steadily as wealth increases.

Conclusion

Based on the above result, it can be concluded here that generalized multivariate semi-Pareto distribution given by Yeh (2007) can be extended to a new family of distributions called Marshall-Olkin generalized multivariate semi-Pareto distribution. Characterization property of generalized multivariate semi-Pareto distribution can be extended to that of Marshall-Olkin generalized multivariate semi-Pareto distribution.

It can also be concluded that the new family of distributions derived in this research work give a fresh result which is more flexible than the distributions stated in Thomas and Jose (2004) as well as Yeh (2007). This is so, since the probability or fraction of population that owns a small amount of wealth per person is rather lower when new family of distributions is used and it continue to reduce as the value of parameter β approaches zero. On the other hand, the probability or fraction of population that owns a small amount of wealth per person attends its peak when parameter $\beta = 1,$ that is, when the distributions given in the literature are used.

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