TRIPLE ACCELERATED OVERRELAXATION METHOD FOR SYSTEM OF LINEAR EQUATIONS

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Abstract
The Accelerated Overrelaxation (AOR) method is basically a stationary iterative method for obtaining solutions for system of linear equations. Such sets of equations can also be solved by employing direct methods. However, iterative methods, particularly the AOR method are mostly suitable when the coefficient matrices A are sparsely large. A new variant of the AOR method designed to speed up convergence of AOR iterative method for solving system of linear equations $Az = b$ is proposed. Simply referred to as Triple Accelerated Overrelaxation (TAOR) method, the method is a three-parameter generalization of the Successive Overrelaxation (SOR) and AOR methods. Convergence conditions of the method for $L-$matrix, $H-$matrix and irreducible matrix with weak diagonal dominance are studied. Numerical samples to demonstrate the efficiency of the method are presented. The findings indicate the superiority of the proposed TAOR method over the AOR method in terms of convergence rate.

Keywords: AOR, convergence, efficiency, generalization, TAOR, three-parameter, Triple, variant

Introduction
Consider a system of linear equations in the matrix form;

$$Az = b \tag{1}$$

A is the matrix of coefficients, b is the column vector of constants on the right hand side, and z is the column vector of the unknowns. If A has a non-vanishing diagonal elements, then a usual splitting of A are obtained thus:

$$A = D - L_A - U_A \tag{2}$$

Where U_A and L_A are the strictly upper and strictly lower components of A respectively, and D is the diagonal part of A. The linear system in (1) can be normalized in such a way that the diagonal elements are all 1s, whereby the matrix A can be expressed as

$$A = I - L - U \tag{3}$$

Hadjidimos (1978) invented the AOR method, a two-parameter generalization of the Jacobi, Gauss-Seidel and SOR methods. By employing the above splitting, the AOR method is governed by the relation

$$z^{[k+1]} = T_{\omega,r} z^{[k]} + (I - rL)^{-1}\omega \dot{b} \tag{4}$$

Where

$$T_{\omega,r} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U\,]$$

The AOR method, viewed as an extrapolation of the SOR, having overrelaxation parameter r and extrapolation parameter $s(= \omega/r)$, was derived through the interpolation procedure with respect to the sub-matrices in application of a general linear stationary schemes (Hadjidimos, 1978; Hadjidimos and Yeyios, 1980). Wu and Liu (2014) presented an alternative form of AOR called Quasi Accelerated Overrelaxation (QAOR) method. This method generalizes the modified Successive Overrelaxation (KSOR) method of Youssef (2012) from the AOR point of view, although it is not as efficient as the AOR and KSOR methods. Youssef and Farid (2015) introduced an extrapolated variant of the KSOR method called the KAOR method. The method is more efficient than the AOR and QAOR due to the fact that they considered the negative values of the relaxation parameters which produces a remarkable outcome to the effect that optimum values occurred in the negative domain. Most recently, Vatti *et al.* (2019) proposed another version of the AOR method called generalized Parametric Accelerated Overrelaxation (PAOR) method, an efficient method designed to solve non-square system of linear equations.

This present work attempts to modify the AOR method in order to improve its effectiveness by introducing one more parameter to its present form.

Derivation of TAOR Method

Consider a general linear stationary iterative method of the form;

$$(\beta_1 D + \beta_2 L_A) z^{[k+1]} = (\beta_3 D + \beta_4 L_A + \beta_5 U_A) z^{[k]} + \beta_6 b \qquad k = 0,1,2,\cdots, \quad (5)$$

where $\beta_i, i = 1,2,\cdots,6$ are constants to be determined $(\beta_1 \neq 0)$ and $z^{(0)}$ an arbitrary initial estimation to the solution z (Hadjidimos, 1978).

$$(D + \beta_2' L_A) z^{(k+1)} = (\beta_3' D + \beta_4' L_A + \beta_5' U_A) z^{(k)} + \beta_6' b, \quad k = 0,1,2,\cdots, \quad (6)$$

where $\beta_i' = \frac{\beta_i}{\beta_1}, i = 2,3,\cdots 6$, $\frac{\beta_1}{\beta_1} = 1$ and for scheme (6) to be consistent with system (1), we proceed from (6) thus

$$[(1-\beta_3')D + (\beta_2' - \beta_4')L_A - \beta_5' U_A]\, z = \beta_6' b, \quad (7)$$

By substituting $z = A^{-1}b$ into (7) and multiplying both sides of the equation by A results in

$$(1-\beta_3')D + (\beta_2' - \beta_4')L_A - \beta_5' U_A \equiv \beta_6' A, \beta_6 \neq 0 \quad (8)$$

Since from (2), $A = D - L_A - U_A$, the following relationships are obtained from (8)

$$1 - \beta_3' = \beta_6', \qquad \beta_2' - \beta_4' = -\beta_6' \text{ and } -\beta_5' = -\beta_6' \quad (9)$$

Equation (9) give the following three– parameter solution of the method:

$$\beta_2' = -(v+r), \qquad \beta_3' = 1-\omega, \qquad \beta_4' = \omega - (v+r), \qquad \beta_5' = \omega, \qquad \beta_6' = \omega \quad (10)$$

Where v, r and $\omega \neq 0$ are any 3 fixed parameters. As a consequence, equation (6) results in

$$[D - (v+r)L_A] z^{[k+1]} = [(1-\omega)D + [\omega - (v+r)]L_A + \omega U_A] z^{[k]} + \omega b \quad (11)$$

Where $L = D^{-1}L_A$, $U = D^{-1}U_A$, $I = D^{-1}D$ and $C = D^{-1}b$.

We obtain

$$[I - (v+r)L] z^{[k+1]} = [(1-\omega)I + [\omega - (v+r)]L + \omega U] z^{[k]} + \omega C, \quad k = 0,1,2,\cdots, \quad (12)$$

Method (11) or (12) is now the proposed Triple Accelerated Overrelaxation iterative method called TAOR. The new parameter v will be called super acceleration parameter, r the acceleration parameter and ω the overrelaxation parameter. The TAOR method can also be written in a more compact form as

$$z^{(k+1)} = T_{v,r,\omega} z^{(k)} + [I - (v+r)L]^{-1} \omega C \quad (13)$$

The notation $T_{v,r,\omega}$ is utilized to represent the TAOR iteration matrix which is represented as

$$T_{v,r,\omega} = [I - (v+r)L]^{-1}[(1-\omega)I + [\omega - (v+r)]L + \omega U] \quad (14)$$

The spectral radius of the TAOR method is the largest eigenvalue of its iteration matrix denoted as $\rho(T_{v,r,\omega})$. The options of $T_{v,r,\omega}$ for $(v,r,\omega) = (0,r,\omega), (0,0,1), (0,0,\omega), (0,1,1), (0,\omega,\omega)$ coincides to methods of AOR, Jacobi, Extrapolated Jacobi, Gauss-Seidel and SOR respectively.

Convergence Theorems

Definition 1: A square matrix A is an L –matrix if $a_{ii} > 0$ and $a_{ij} \leq 0$, $i \neq j$ for all $i, j = 1,2,\cdots,N$.

Definition 2: A matrix $A = [a_{ij}]$ is irreducible if and only if its directed graph $G(A)$ is strongly connected.

Definition 3: A square irreducible matrix A is said to be weakly diagonally dominant if

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad i = 1, 2, \cdots, n$$

Definition 4: Let a matrix $A = (a_{i,j})$ be a nonsingular matrix and define its comparison matrix as $C\langle A\rangle = \begin{cases} -|a_{i,j}|, & \text{when } i \neq j \\ |a_{i,i}|, & \text{when } i = j \end{cases}$, then A is an $H-$matrix if $C\langle A\rangle$ is a $M-$matrix.

Lemma 1 (Varga, (2000))

Let $A \geq 0$ be an irreducible square matrix. Then

i. A has a positive real eigenvalue equal to its spectral radius.
ii. To the spectral radius of A denoted as $\rho(A)$, there corresponds an eigenvector $z > 0$.
iii. $\rho(A)$ increases when any entry of A increases.
iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 (Young, (2014))

Suppose A and B are two matrices of compatible size with $\rho(A)$ representing the spectral radius of A and $\rho(B)$ representing the spectral radius of B, if $|A| \leq B$ implies that $\rho(A) \leq \rho(B)$, where $|A|$ represents the moduli of the corresponding elements of A.

Theorem 1: If matrix A is an $L-$matrix, for all v, r and ω such that $0 \leq v + r \leq \omega \leq 1$ and $\omega \neq 0$, then the TAOR iterative method converges for $\rho(T_{0,0,1}) < 1$.

This theorem will be established using similar ideas from works of Youssef and Farid (2015), Wu and Liu (2014) and Hadjidimos (1978).

Proof: Suppose that $\dot{\lambda} = \rho(T_{v,r,\omega}) \geq 1$, with the assumption in mind, we can easily obtain

$$[I - (v + r)L]^{-1} = I + (v + r)L + (v + r)^2L^2 + (v + r)^3L^3 + \cdots + (v + r)^{N-1}L^{N-1} \geq 0 \tag{15}$$

Therefore, for the matrix of iteration, it gives

$$T_{v,r,\omega} = [I - (v + r)L]^{-1}[(1 - \omega)I + [\omega - (v + r)]L + \omega U] \geq 0 \tag{16}$$

Implying that $T_{v,r,\omega}$ is a nonnegative matrix then by lemma 1, it follows that $\dot{\lambda}$ is an eigenvalue of $T_{v,r,\omega}$. If $z \neq 0$ is the eigenvector corresponding to $\dot{\lambda}$, then it implies that $T_{v,r,\omega}\, z = \dot{\lambda} z$ which yields the following equations below;

$$[I - vL - rL]^{-1}[(1 - \omega)I + [\omega - (v + r)]L + \omega U]z = \dot{\lambda} z$$

$$\left[\frac{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)}{\omega} L + U\right] z = \left(\frac{\dot{\lambda} + \omega - 1}{\omega}\right) Iz \tag{17}$$

Equation (17) indicates that $\frac{\dot{\lambda}+\omega-1}{\omega}$ is an eigenvalue of $\frac{\omega+r(\dot{\lambda}-1)+v(\dot{\lambda}-1)}{\omega} L + U$, hence (17) yields

$$\frac{\dot{\lambda} + \omega - 1}{\omega} \leq \rho\left(\frac{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)}{\omega} L + U\right) \tag{18}$$

It is easily seen that $\frac{\omega+r(\dot{\lambda}-1)+v(\dot{\lambda}-1)}{\omega} \geq 1$ so that

$$0 \leq \frac{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)}{\omega} L + U \leq \frac{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)}{\omega} (L + U)$$

$$= \frac{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)}{\omega} T_{0,0,1} \tag{19}$$

Where $T_{0,0,1}$ is the Jacobi matrix and combining equations (18) with (19), yields

$$\dot{\lambda} + \omega - 1 \leq \omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)\rho(T_{0,0,1}) \quad (20)$$

Simple manipulation on (20) results into

$$\rho(T_{0,0,1}) \geq \frac{\dot{\lambda} + \omega - 1}{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)} \geq 1 \quad (21)$$

And thus we obtain

$$\rho(T_{0,0,1}) \geq 1 \quad (22)$$

Now, suppose $\rho(T_{0,0,1}) < 1$, then it becomes

$$\frac{\dot{\lambda} + \omega - 1}{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)} < 1 \quad (23)$$

Which implies $\dot{\lambda} < 1$, so that if $\rho(T_{0,0,1}) < 1$ then the TAOR method equally converges {that is $\rho(T_{v,r,\omega}) < 1$} since the spectral radius of the Jacobi matrix { $\rho(T_{0,0,1})$} is incorporated inside the TAOR iterative method. Hence, the theorem is completed and proved.

Theorem 2: If A is an irreducible matrix with weak diagonal dominance, then the TAOR iterative method converges for all $0 \leq v + r \leq 1$ and $0 < \omega \leq 1$.

We shall employ the ideas of authors like Hadjidimos (1978), Wu and Liu (2014) and Youssef and Farid (2015) to prove the theorem.

Proof: Let A be an irreducible matrix, for some eigenvalue $\dot{\lambda}$ of $T_{v,r,\omega}$, we assumed that $|\dot{\lambda}| \geq 1$. For this $\dot{\lambda}$, the relationship in (24) holds

$$\det(T_{v,r,\omega} - \dot{\lambda}I) = 0 \quad (24)$$

Simple transformation was performed on (24) to obtain

$$\det(\mathrm{R}) = 0 \quad (25)$$

Where R is given as

$$\mathrm{R} = I - \frac{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)}{\dot{\lambda} + \omega - 1}L - \frac{\omega}{\dot{\lambda} + \omega - 1}U \quad (26)$$

The moduli of the coefficients of L and U in (26) are less than one. To prove this, it is sufficient and necessary to prove that

$$|\dot{\lambda} + \omega - 1| \geq |\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)| \quad \text{and} \quad |\dot{\lambda} + \omega - 1| \geq |\omega| \quad (27)$$

suppose $\dot{\lambda}^{-1} = qe^{i\theta}$, where q and θ are real with $0 < q \leq 1$, then the first inequality in (27) is equivalent to;

$$[1 - (v + r)^2] + [1 - (v + r)^2]q^2 - [1 - (v + r)^2]2qcos\theta + [1 - (v + r)]2q\omega cos\theta - [1 - (v + r)]2q^2\omega \geq 0 \quad (28)$$

Which holds for $v + r = 1$, otherwise it is equivalent to;

$$(1 + v + r) + (1 + v + r)q^2 - [(1 + v + r) - \omega]2qcos\theta - 2q^2\omega \geq 0 \quad (29)$$

Since the expression in the brackets above is nonnegative, (29) holds for all θ if and only if it holds for $cos\theta = 1$, hence (29) is equivalent to

$$(1 - q)[(1 + v + r)(1 - q)] + 2q\omega \geq 0 \quad (30)$$

Which is true, similarly, the second inequality in (27) is equivalently

$$1 + q^2 - 2q(1 - \omega)cos\theta - 2\omega q^2 \geq 0 \quad (31)$$

Which for same reason, must be satisfied for $cos\ \theta = 1$, then equation (31) is equivalent to

$$(1 - q)[1 - q + 2\omega q] \geq 0 \quad (32)$$

Which holds for $q = 1$, thus we have

$$\left|\frac{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)}{\dot{\lambda} + \omega - 1}\right| \leq 1 \quad \text{and} \quad \left|\frac{\omega}{\dot{\lambda} + \omega - 1}\right| \leq 1 \quad (33)$$

Given that A is irreducible with a weak diagonal dominance, it signifies that $D^{-1}A = I - L - U$ equally have the same properties too. This is also true for R, considering that 1 is greater than the modulus of the coefficients of L and U and they differs from zero. Hence it implies that $\det(\mathrm{R}) \neq 0$ which contradicts (25) and consequently (24). Hence $\rho(T_{v,r,\omega}) < 1$ impying that the TAOR method converges and this completes the proof.

Theorem 3: If the coefficient matrix A is an $H-$matrix with the domain $0 \leq v + r \leq \omega \leq 1$, and $\omega \neq 0$, then the TAOR method converges to the true solution for any initial guess $z^{(0)}$.
Proof: Let A be an $H-$ matrix with splitting of A into $I - L - U$ and the regular splitting of A into $\omega A = M - N$ by the proposed TAOR method having choices $M = (I - (v + r)L)$ and $N = (1 - \omega)I + (\omega - v - r)L + \omega U$, then the comparison matrix $S\langle A\rangle = I - |L| - |U|$, with choices of $M_{S\langle A\rangle}$ and $N_{S\langle A\rangle}$ is obtained as

$$M_{S\langle A\rangle} = (I - (v + r)|L|), \qquad N_{S\langle A\rangle} = \big((1 - \omega)|I| + (\omega - v - r)|L| + \omega|U|\big) \quad (34)$$

Then one can obtain

$$\begin{aligned}|(I - (v + r)L)^{-1}| &= |I + (v + r)L + (v + r)^2L^2 + (v + r)^3L^3 + \cdots + (v + r)^{N-1}L^{N-1}| \\ &\leq (I + (v + r)|L| + (v + r)^2|L|^2 + (v + r)^3|L|^3 + \cdots + (v + r)^{N-1}|L|^{N-1}) \\ &\leq (I - (v + r)|L|)^{-1} = J^{-1} \qquad (35)\end{aligned}$$

Also taking the modulus of the matrix N to obtain

$$\begin{aligned}|N| &= |(1 - \omega)I + (\omega - v - r)L + \omega U| \\ &= \big((1 - \omega)I + (\omega - v - r)|L| + \omega|U|\big) \\ &\leq (1 - \omega)I + (\omega - v - r)|L| + \omega|U| = R\end{aligned} \qquad (36)$$

This implies that $|N| \leq R$, and similarly, the modulus of matrix $T_{v,r,\omega}$ gives

$$\begin{aligned}|T_{v,r,\omega}| &= \left|(I - (v + r)L)^{-1} \times \big((1 - \omega)I + (\omega - v - r)L + \omega U\big)\right| \\ &= |(I + (v + r)L + (v + r)^2L^2 + (v + r)^3L^3 + \cdots + (v + r)^{N-1}L^{N-1} \\ &\times [(1 - \omega)I + [\omega - (v + r)]L + \omega U\ ])| \\ &\leq (1 - \omega)I + (1 - \omega)(v + r)|L| + (1 - \omega)(v + r)^2|L|^2 + (\omega - v - r)|L| \\ &+ (v + r)(\omega - v - r)|L|^2 + (v + r)^2(\omega - v - r)|L|^3 + \omega + \omega(v + r)|L||U| \\ &+ \omega(v + r)^2|L|^2|U| + \cdots \\ &\leq [\mathrm{I} - (v + r)|L|]^{-1} \times \big((1 - \omega)I + (\omega - v - r)|L| + \omega|U|\big) = J^{-1}R \qquad (37)\end{aligned}$$

Obviously, it is shown that $|T_{v,r,\omega}| \leq J^{-1}R$, since it holds, then applying lemma 2 to it implies that

$$\rho(T_{v,r,\omega}) \leq \rho(J^{-1}R) \qquad (38)$$

And $\rho(J^{-1}R) < 1$ if and only if $\omega[I - |B|]$ is a monotone matrix, that is

$$J - R = I - (v + r)|L| - [(1 - \omega)I + (\omega - v - r)|L| + \omega|U|] = \omega[I - |B|] \qquad (39)$$

Given that A is an H matrix, then $\omega[I - |B|]$ with $\omega > 0$ is a monotone matrix, therefore the TAOR iterative method converges for $H-$ matrices and the theorem is completed. The idea of the above proof is obtained from similar proofs of Wu and Liu (2014) and Yun (2008).

Numerical Samples and Discussion

Numerical Samples

Sample 1: Considering the linear system $Az = b$ from Ndanusa (2012).

$$\begin{bmatrix} \frac{43}{9} & -\frac{4}{3} & -\frac{10}{9} & 0 \\ -\frac{5}{3} & \frac{49}{9} & 0 & -\frac{10}{9} \\ -\frac{13}{9} & 0 & \frac{49}{9} & -\frac{4}{3} \\ 0 & -\frac{13}{9} & -\frac{5}{3} & \frac{55}{9} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} \\ \frac{8}{27} \\ \frac{22}{9} \\ \frac{62}{27} \end{bmatrix} \tag{40}$$

The solutions of (40) for $v = 0.5,\ r = 0.4$ and $\omega = 1.0$, with tolerance of 10 decimal places is presented in table 1

Sample 2: Consider the system of linear equations expressed in matrix form $Az = b$

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \end{pmatrix} = \begin{pmatrix} 1500 \\ 1000 \\ 1000 \\ 2000 \\ 0 \\ 0 \\ 1500 \\ 1000 \\ 1000 \end{pmatrix} \tag{41}$$

The numerical solution of (41) is obtained using methods of AOR and TAOR for values $v = 0.5$, $r = 0.4$ and $\omega = 1.0$, results tabulated in table 2

Sample 3: In this sample, we consider the system of linear equations from Mohammed and Rivaie (2017) in the form $Az = b$

$$\begin{pmatrix} 7 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 7 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 7 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 7 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 7 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 7 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 7 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 7 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 7 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 7 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} \tag{42}$$

(42) is solved with methods of AOR and TAOR for the same values of parameters in sample 1 using maple 2017 software. The results are displayed in table 3.

Table 1: Numerical Result of AOR and TAOR Methods for sample 1

No of Iterations	z_k	AOR	TAOR
1	z_1	0.1162790698	0.1162790698
	z_2	0.0686600221	0.0864578389
	z_3	0.4613194115	0.4767441860
	z_4	0.4325748227	0.5111685435

2	z_1	0.2427235902	0.2512775797
	z_2	0.1937809809	0.2315309803
	z_3	0.5991844344	0.6372476122
	z_4	0.5446701318	0.5964715158
⋮	⋮	⋮	⋮
22	z_1	0.3615974117	0.3615974257
	z_2	0.2950388033	0.2950388157
	z_3	0.7008223394	0.7008223517
	z_4	0.6366273808	0.6366273918
⋮	⋮	⋮	
32	z_1	0.3615974257	
	z_2	0.2950388157	
	z_3	0.7008223517	
	z_4	0.6366273918	

Table 2: **Numerical Result of AOR and TAOR Methods for sample 2**

No of Iterations	z_k	AOR	TAOR
1	z_1	375.0000000000	375.0000000000
	z_2	287.5000000000	334.3750000000
	z_3	278.7500000000	325.2343750000
	z_4	537.5000000000	584.3750000000
	z_5	82.5000000000	206.7187500000
	z_6	36.1250000000	119.6894531250
	z_7	428.7500000000	506.4843750000
	z_8	301.1250000000	410.4707031250
	z_9	283.7250000000	369.2860351562
2	z_1	581.2500000000	604.6875000000
	z_2	454.6875000000	528.4179687500
	z_3	347.6250000000	407.1757812500
	z_4	742.1875000000	823.7304687500
	z_5	327.7500000000	459.7421875000
	z_6	192.6562500000	300.6768798828
	z_7	605.1250000000	677.5664062500
	z_8	490.9062500000	616.0460205078
	z_9	368.9437500000	469.5166564941
⋮	⋮	⋮	⋮

56	z_1	861.6071428185	861.6071428571
	z_2	812.4999999492	812.5000000000
	z_3	575.8928571095	575.8928571429
	z_4	1133.9285713778	1133.9285714286
	z_5	812.4999999333	812.5000000000
	z_6	491.0714285276	491.0714285714
	z_7	861.6071428238	861.6071428571
	z_8	812.4999999562	812.5000000000
	z_9	575.8928571141	575.8928571429
⋮	⋮	⋮	
81	z_1	861.6071428571	
	z_2	812.5000000000	
	z_3	575.8928571429	
	z_4	1133.9285714286	
	z_5	812.5000000000	
	z_6	491.0714285714	
	z_7	861.6071428571	
	z_8	812.5000000000	
	z_9	575.8928571429	

Table 3: Numerical Result of AOR and TAOR Methods for sample 3

No of Iterations	z_k	AOR	TAOR
1	z_1	1.0000000000	1.0000000000
	z_2	0.4857142857	0.5571428571
	z_3	0.3134693877	0.3573469388
	z_4	0.3607696793	0.4602303207
	z_5	0.3340847980	0.4165194086
	z_6	0.3798602392	0.5137828161
	z_7	0.3557910974	0.4825771992
	z_8	0.4001911590	0.5758284560
	z_9	0.3786591636	0.5566122864
	z_{10}	0.4218288255	0.6473928928
2	z_1	1.2323621947	1.3009977786
	z_2	0.7821356178	0.8691362620
	z_3	0.5952761031	0.7193102010
	z_4	0.6553817158	0.7728172528
	z_5	0.6121110766	0.7594999494
	z_6	0.6712689317	0.8169147509
	z_7	0.6287630019	0.7984740553
	z_8	0.6868673263	0.8575300610
	z_9	0.6451444972	0.8346928331
	z_{10}	0.7020950596	0.8932832741
⋮	⋮	⋮	⋮

43	z_1	1.6995918265	1.6995918367
	z_2	1.3314285592	1.3314285714
	z_3	1.1551020286	1.1551020408
	z_4	1.1885714167	1.1885714286
	z_5	1.1551020289	1.1551020408
	z_6	1.1885714171	1.1885714286
	z_7	1.1551020293	1.1551020408
	z_8	1.1885714174	1.1885714286
	z_9	1.1551020296	1.1551020408
	z_{10}	1.1885714178	1.1885714286
⋮	⋮	⋮	
60	z_1	1.6995918367	
	z_2	1.3314285714	
	z_3	1.1551020408	
	z_4	1.1885714286	
	z_5	1.1551020408	
	z_6	1.1885714286	
	z_7	1.1551020408	
	z_8	1.1885714286	
	z_9	1.1551020408	
	z_{10}	1.1885714286	

Discussion

Three numerical samples of 4×4, 9×9 and 10×10 linear systems with accuracy of 10 decimal places were illustrated using Maple 2017 software. The results of TAOR method obtained are compared with that of AOR method. The new method (TAOR) converges at 22nd, 56th and 43rd iterations in comparison to 32nd, 81st and 60th iterations of the AOR method for sample 1, 2 and 3 respectively.

Conclusion

In this research work, a new improved Accelerated Over-relaxation method has been proposed, namely the TAOR method. Convergence criteria for some special matrices were examined and the proposed method was compared with AOR method by rate of convergence. Outcome of the findings reveals that the new improved AOR method performs much better than the existing AOR method. Also, the study has shown that introduction of another acceleration parameter in the AOR scheme increases convergence of the new method.

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References

Hadjidimos, A., & Yeyious, A. (1980). The principle of extrapolation in connection with the accelerated overrelaxation method. *Linear Algebra and its Application,* 30, 115-128.

Hadjidimos, A. (1978). Accelerated overrelaxation method. *Mathematics of Computation,* (32), 149 -157.

Mohammed, F. D., & Rivaie, M. (2017). Jacobi-Davisson, Gauss-seidel and successive over-relaxation for solving systems of linear equations. *Applied Mathematics and Computational Intelligence*, 6, 41-52.

Ndanusa, A. (2012). Preconditioned successive over-relaxation iterative method for solving partial equations of elliptic type. Doctoral Thesis, Federal University of Technology, Minna, Nigeria.

Varga, R. S. (2000). *Matrix iterative analysis*. Berlin, Germany: Springer.

Vatti, V. B. K., Kumar, M. S., & Rao, G. C. (2019). Generalized PAOR method for solving non-square linear systems. *Compliance Engineering Journal*, 10, 77-74.

Wu, S., & Liu, Y. (2014). A new version of accelerated overrelaxation iterative method. *Journal of Applied Mathematics,* (725360), 1-6.

Young, D. M. (2014). *Iterative Solutions of Large Linear Systems*, Elserier.

Youssef, I. K. (2012). On the successive overralaxation method. *Journal of Mathematics and Statistics*, 8(2); 176-184.

Youssef, I. K., & Farid, M. M. (2015). On the accelerated overrelaxation method. *Pure and Applied Mathematics Journal,* 4, 26 – 31.

Yun, J. H. (2008). Convergence of SSOR multisplitting methods for an $H-$matrix. *Journal of Computational and Applied Mathematics,* Elsevier, 217, 252-258.