HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF MULTIVALENT FUNCTIONS DEFINED BY SALAGEAN DIFFERENTIAL OPERATOR

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Abstract
The paper obtains the sharp bounds for the second Hankel determinant of Certain Subclasses of Multivalent functions defined by Salagean differential operator.

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Introduction and Preliminaries

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad p \in N \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk $U = \{z : |z| < 1\}$.

Opoola, in 1984 defined the class $T_n^{\beta}(\alpha)$ to be the subclass of the functions defined in (1.1) when $p = 1$, satisfying the condition

$$\operatorname{Re}\left\{\frac{D^n f(z)^{\beta}}{z^{\beta}}\right\} > \alpha \;\; z \in U,\; n \in N,\; 0 \le \alpha < 1, \beta \ge 0 \tag{1.2}$$

where D^n is the Salagean differential operator defined recursively as follows:

$$\left.\begin{aligned} &D^0 f(z) = f(z) \\ &D^1 f(z) = Df(z) = zf'(z) \\ &D^2 f(z) = D(D^1 f(z)) = z(D^1 f(z))' \\ &\vdots \\ &D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))' \end{aligned}\right\} \tag{1.3}$$

The class has been repeatedly investigated by several authors (see Ntatin (2010), Al-Shaqsi et al (2010) and Fadipe-Joseph et al (2007)).

From (1.1), and using binomial expansion we have

$$\left.\begin{aligned} f(z)^{\beta} &= z^{p\beta} + \beta a_{p+1} z^{p\beta+1} + \left(\beta a_{p+2} + \frac{\beta(\beta-1)}{2!} a_{p+1}^2\right) z^{p\beta+2} \\ &+ \left(\beta a_{p+3} + \frac{\beta(\beta-1)}{2!} 2a_{p+1}a_{p+2} + \frac{\beta(\beta-1)(\beta-2)}{3!} a_{p+1}^3\right) z^{p\beta+3} + \cdots \end{aligned}\right\} \tag{1.4}$$

which implies

$$f(z)^\beta = z^{p\beta} + \sum_{k=1}^{\infty} \beta a_{p+k} z^{\beta p+k} \qquad \beta > 0 \tag{1.5}$$

And by the operator defined in (1.3), (1.5) becomes

$$D^n f(z)^\beta = z^{p\beta} + \sum_{k=1}^{\infty} \beta \left(\frac{p\beta + k}{p\beta}\right)^n a_{p+k} z^{\beta p+k} \tag{1.6}$$

From (1.2) and (1.6) we defined the class of functions $f(z) \in A_p$ satisfying the condition

$$Re\left\{\frac{D^n f(z)^\beta}{z^{p\beta}}\right\} > \alpha, z \in U, p, n \in N, 0 \le \alpha < p, \beta \ge 0 \tag{1.7}$$

The class is denoted by $T_n^\beta(\alpha, p)$.

Similarly a function $f(z) \in A_p$ is in the class $T_n^\beta(\lambda, p)$ if it satisfies the condition

$$\mathrm{Re}\left(\frac{D^n f(z)^\beta}{(1-\lambda) f(z)^\beta + \lambda D^n f(z)^\beta}\right) > \lambda \ z \in U, \ p, n \in N, \ 0 \le \lambda < p, \ \beta \ge 0 \tag{1.8}$$

The Hankel determinants $H_q(n)$ of $f(z)$ for $q \ge 1$ and $n \ge 1$ stated in Vamshee and Ramreddy (2016), Yavuz 2015 and Noonan and Thomas (1976) is defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \ (a_1 = 1)$$

The determinant has been considered by different authors for several classes of functions in geometric function theory (see, Sahoo (2018), Alarifi et al. (2017), Patil and Khairnar (2017), Vamshee and Ramreddy (2016). Yavuz (2015). Sudharsan and Vijaya (2014)).

Specifically for the case of $q = 2$ and $n = 2$ known as the second Hankel determinant (functional), given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$$

were obtained for various subclasses of univalent and multivalent analytic functions. The bounds give information about the geometric properties of this class of functions, in particular, the growth and distortion properties of these functions are determined by the bounds of its second coefficient. Vamshee and Ramreddy (2016) considered the Hankel determinant for the case of $q = 2$ and $n = p+1$ denoted by $H_2(p+1)$, given by

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1} a_{p+3} - a_{p+2}^2$$

for the class of Multivalent functions of bounded turning of order alpha. And obtain the sharp upper bound for the functional $\left|a_{p+1} a_{p+3} - a_{p+2}^2\right|$ i.e.

$$\left|a_{p+1}a_{p+3}-a_{p+2}^{2}\right|\le\left[\frac{2p(1-\alpha)}{p+2}\right]^{2}$$

for the class

Motivated by this result and several others, we consider the second Hankel determinant for the class of functions defined in (1.7) and (1.8), and seek a sharp upper bound for the functional $\left|a_{p+1}a_{p+3}-a_{p+2}^{2}\right|$ for functions belonging to this classes.

Preliminary results

The following Lemmas are required to prove our results. Let P be the class of all functions $p(z)$ analytic in the unit disk U such that $\operatorname{Re} p(z)>0$ and

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots=1+\sum_{k=1}^{\infty}c_kz^k \qquad (2.1)$$

Lemma 2.1: (Pommerenke, 1966).

Let $p\in P$ then $|c_k|\le 2,\ k=1,2,\cdots$ and the inequality is sharp for

$$p(z)=\frac{1+z}{1-z}=1+\sum_{k=1}^{\infty}2z^k \qquad (2.2)$$

Lemma 2.2: (Hayami & Owa, 2009)

If $p(z)=p+\sum_{k=1}^{\infty}c_kz^k$ satisfies the following inequality

$$\operatorname{Re} p(z)>\alpha(z\in U)$$

for some $\alpha(0\le\alpha<p)$, then

$$|c_k|\le 2(p-\alpha)\ (k=1,2,3,\cdots). \qquad (2.3)$$

Lemma 2.3: (Libera & Zlotkiewicz, 1983)

The power series for $p(z)$ given in (2.1) converges in U to a function in P if and only if the Toeplitz determinant

$$D_n=\begin{vmatrix}2 & c_1 & c_2 & \dots & c_n\\ c_{-1} & 2 & c_1 & \cdots & c_{n-1}\\ c_{-2} & c_{-1} & 2 & \dots & c_{n-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ c_n & c_{-n+1} & c_{-n+2} & \dots & 2\end{vmatrix},\ n=1,2,3,\cdots \qquad (2.5)$$

and $c_{-k}=\overline{c}_k$ are all non-negative. They are strictly for $p(z)=\sum_{k=1}^{m}p_kp_0(e^{it_k}z)$, with $p(z)=\sum_{k=1}^{m}p_k=1$, t_k real and $t_k\ne t_j$, for $k\ne j$, where $p_0(z)=\frac{1+z}{1-z}$ in this case $D_n>0$ for $n<(m-1)$ and $D_n=0$ for $n\le m$.

Lemma 2.4: (Hayami & Owa, 2009)

If a function $p(z)=p+\sum_{k=1}^{\infty}c_k z^k$ satisfies $\operatorname{Re} p(z)>\alpha (z\in U)$ for some $\alpha(0\le\alpha<p)$, then

$$2(p-\alpha)c_2=c_1^2+\{4(p-\alpha)^2-c_1^2\}x$$

$$\left.\begin{aligned}&4(p-\alpha)^2c_3=c_1^3+2\{4(p-\alpha)^2-c_1^2\}c_1x-\{4(p-\alpha)^2-c_1^2\}c_1x^2\\&+2(p-\alpha)\{4(p-\alpha)^2-c_1^2\}(1-|x|^2)y\end{aligned}\right\} \quad (2.6)$$

for some complex number x *and* y $(|x|\le 1,|y|\le 1)$

Main Results

Theorem 3.1:

If $f(z)\in T_n^{\beta}(\alpha,p)$, then

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right|\le\frac{4(p-\alpha)\beta^n}{(p\beta)^n(p\beta+2)^{2n}} \quad (3.1)$$

The result obtained is sharp.

Proof:

Let $f(z)\in T_n^{\beta}(\alpha,p)$. Then there exist a $p(z)\in P$, such that

$$\frac{D^n f(z)^{\beta}}{z^{p\beta}}=p(z). \quad (3.2)$$

Expanding both side of (3.2) we have

$$1+\beta\left(\frac{p\beta+1}{p\beta}\right)^n a_{p+1}z+\beta\left(\frac{p\beta+2}{p\beta}\right)^n a_{p+2}z^2+\beta\left(\frac{p\beta+3}{p\beta}\right)^n a_{p+3}z^3+\cdots$$

$$=1+c_1z+c_2z^2+c_3z^3+\cdots, \quad (3.4)$$

equating powers of z, z^2 and z^3 we obtain

$$a_{p+1}=\frac{c_1(p\beta)^n}{\beta(p\beta+1)^n},\ a_{p+2}=\frac{c_2(p\beta)^n}{\beta(p\beta+2)^n},\ a_{p+3}=\frac{c_3(p\beta)^n}{\beta(p\beta+3)^n} \quad (3.5)$$

from (3.3) and (3.4) we have that

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right|=\left|\frac{c_1(p\beta)^n}{\beta(p\beta+1)^n}\frac{c_3(p\beta)^n}{\beta(p\beta+3)^n}-\left(\frac{c_2(p\beta)^n}{\beta(p\beta+2)^n}\right)^2\right|$$

$$=\frac{(p\beta)^{2n}}{\beta^2}\left|\frac{c_1c_3}{(p\beta+1)^n(p\beta+3)^n}-\frac{c_2^2}{(p\beta+2)^{2n}}\right|$$

substituting for c_2 and c_3 using lemma 2.4, and for some x and y such that $|x|\le 1,\ |y|\le 1.$ simplifying we have

$$=\left|\begin{array}{l}\left(\dfrac{c_1^4+2[4(p-\alpha)^2-c_1^2]c_1x-[4(p-\alpha)^2-c_1^2]c_1x^2+2(p-\alpha)[4(p-\alpha)^2-c_1^2](1-|x|^2)y}{4(p-\alpha)^2(p\beta+1)^n(p\beta+3)^n}\right)\\ -\left(\dfrac{c_1^4+2[4(p-\alpha)^2-c_1^2]c_1x+[(4(p-\alpha)^2-c_1^2)x]^2}{4(p-\alpha)^2(p\beta+2)^{2n}}\right)\end{array}\right|$$

If $c_1=c$ and $0\le c\le 2(p-\alpha)$, applying the triangle inequality we obtain

$$\left.\begin{array}{l}|a_{p+1}a_{p+3}-a_{p+2}^2|\le \dfrac{c^4}{4(p-\alpha)^2(p\beta+1)^n(p\beta+3)^n}+\dfrac{2[(4(p-\alpha)^2-c^2]c\rho-[(4(p-\alpha)^2-c^2]c\rho^2}{4(p-\alpha)^2(p\beta+1)^n(p\beta+3)^n}\\ +\dfrac{8c(p-\alpha)^3-8c(p-\alpha)^3\rho^2}{4(p-\alpha)^2(p\beta+1)^n(p\beta+3)^n}-\dfrac{2c^4+2c^4\rho^2}{4(p-\alpha)^2(p\beta+1)^n(p\beta+3)^n}\\ +\dfrac{c^4}{4(p-\alpha)^2(p\beta+2)^n}+\dfrac{2[(4(p-\alpha)^2-c^2]c\rho-[(4(p-\alpha)^2-c^2]c\rho^2}{4(p-\alpha)^2(p\beta+2)^n}=F'(\rho)\end{array}\right\} \quad (3.6)$$

with $\rho=|x|\le 1$. Furthermore,

$$F(\rho)=\frac{2[(4(p-\alpha)^2-c^2]c\rho-[(4(p-\alpha)^2-c^2]c\rho^2}{4(p-\alpha)^2(p\beta+1)^n(p\beta+3)^n}+\frac{8c(p-\alpha)^3-8c(p-\alpha)^3 2\rho}{4(p-\alpha)^2(p\beta+1)^n(p\beta+3)^n}$$
$$-\frac{4\rho c^4}{4(p-\alpha)^2(p\beta+1)^n(p\beta+3)^n}+\frac{2[(4(p-\alpha)^2-c^2]c-[(4(p-\alpha)^2-c^2]2c\rho}{4(p-\alpha)^2(p\beta+2)^n}$$

and by elementary calculus we see that $F'(\rho)>0$ for $\rho>0$. Thus implying that F is an increasing function and the upper bound for (3.6) correspond to $\rho=1$ and $c=0$ which gives,

$$|a_{p+1}a_{p+3}-a_{p+2}^2|\le\frac{4(p-\alpha)\beta^n}{(p\beta)^n(p\beta+2)^{2n}} \quad (3.7)$$

This complete the proof

Remark 1: For $\alpha=0$ *and* $p=\beta=n=1$ then

$$|a_2a_4-a_3^2|\le\frac{4}{9}$$

Which coincide with the results of Vamshee and Ramreddy (2016) and Janteg et al. (2007)

Remark 2: For $p=2$ *and* $\alpha=\beta=n=1$ then

$$|a_2a_4-a_3^2|\le\frac{1}{8}$$

Which coincide with the results of Yavuz (2015)

Theorem 3.2:

If $f(z) \in T_n^\beta(\lambda, p)$, then

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \frac{4(p-\alpha)^2}{\beta^2(1-\lambda)^2\left[\left(\frac{p\beta+2}{p\beta}\right)^2 - 1\right]^2} \tag{3.8}$$

The result obtained is sharp.

Proof:

Let $f(z) \in T_n^\beta(\lambda, p)$. Then there exist a $p(z) \in P$ with $p(0)=1$ and $\operatorname{Re} p(z) > 0 \in U$, such that

$$\frac{D^n f(z)^\beta}{(1-\lambda)f(z)^\beta + \lambda D^n f(z)^\beta} = p(z) \tag{3.9}$$

Expanding both side of (3.9) and equating the powers of z, z^2 and z^3 gives

$$a_{p+1} = \frac{c_1}{d_{\lambda 1}},$$

$$a_{p+2} = \frac{[\beta + (1+\lambda)c_1 d_1]c_1 + c_2 d_{\lambda 1}}{d_{\lambda 1} d_{\lambda 2}}$$

and

$$a_{p+3} = \frac{[\beta(1-\lambda) + \lambda\beta d_2][\beta(1+\lambda)d_1 c_1]c_1^2 + c_2 d_1 + \beta(1-\lambda)c_2 c_1 + c_3 d_{\lambda 1}}{d_{\lambda 1} d_{\lambda 2} d_{\lambda 3}} \tag{3.11}$$

Where

$$\left.\begin{aligned} d_{\lambda 1} &= \beta(1-\lambda)\left[\left(\frac{p\beta+1}{p\beta}\right)^n - 1\right] \\ d_{\lambda 2} &= \beta(1-\lambda)\left[\left(\frac{p\beta+2}{p\beta}\right)^n - 1\right] \\ d_{\lambda 3} &= \beta(1-\lambda)\left[\left(\frac{p\beta+3}{p\beta}\right)^n - 1\right] \end{aligned}\right\} \tag{3.11}$$

with $d_1 = \left(\frac{p\beta+1}{p\beta}\right)^n$ and $d_2 = \left(\frac{p\beta+2}{p\beta}\right)^n$ respectively

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| = \left|\frac{[\beta(1-\lambda) + \lambda\beta d_2][\beta(1+\lambda)d_1 c_1]c_1^3 + (d_1 + \beta(1-\lambda))c_2 c_1^2 + c_3 c_1 d_{\lambda 1}}{d_{\lambda 1} d_{\lambda 2} d_{\lambda 3}} - \left[\frac{[\beta + (1+\lambda)c_1 d_1]c_1 + c_2 d_{\lambda 1}}{d_{\lambda 1} d_{\lambda 2}}\right]^2\right| \tag{3.12}$$

Applying triangle inequality, substituting for c_2 and c_3 by lemma 2.4, and letting $c_1 = c$ for some x and y such that $|x| \le 1,\ |y| \le 1$. simplifying gives

$$\left.\begin{aligned} |a_{p+1}a_{p+3} - a_{p+2}^2| &\le \frac{[\beta(1-\lambda)+\lambda\beta d_2][\beta(1+\lambda)d_1 c]c^3}{d_{\lambda_1}^2 d_{\lambda_2} d_{\lambda_3}} + \frac{[\beta(1-\lambda)cd_1]^2 c^2}{d_{\lambda_1}^2 d_{\lambda_2}^2} \\ &+ \frac{(d_1+\beta(1-\lambda))(c_1^4 + \{4(p-\alpha)^2 - c^2\}\rho c^2}{d_{\lambda_1}^2 d_{\lambda_2} d_{\lambda_3} 2(p-\alpha)} \\ &+ \frac{c^4 + 2\{4(p-\alpha)^2 - c^2\}c^2\rho - \{4(p-\alpha)^2 - c^2\}c^2\rho^2 + 2c(p-\alpha)\{4(p-\alpha)^2 - c^2(1-\rho^2)}{d_{\lambda_1}^2 d_{\lambda_2} d_{\lambda_3} 4(p+\alpha)^2} \\ &+ \frac{2[\beta+(1+\lambda)cd_1](c^3 + \{4(p-\alpha)^2 - c^2\}\rho c)}{d_{\lambda_1} d_{\lambda_2}^2 2(p-\alpha)} + \frac{[(c^2 + \{4(p-\alpha^2) - c^2\}\rho)]^2}{d_{\lambda_2}^2 4(p-\alpha)} \end{aligned}\right\} \quad (3.13)$$

with $\rho = |x| \le 1$. Furthermore,

$$\left.\begin{aligned} |a_{p+1}a_{p+3} - a_{p+2}^2| &\le \frac{(d_1+\beta(1-\lambda))(c_1^4 + \{4(p-\alpha)^2 - c^2\}c^2}{d_{\lambda_1}^2 d_{\lambda_2} d_{\lambda_3} 2(p-\alpha)} \\ &+ \frac{c^4 + 2\{4(p-\alpha)^2 - c^2\}c^2 - \{4(p-\alpha)^2 - c^2\}c^2 2\rho + 2c(p-\alpha)\{4(p-\alpha)^2 - c^2(1-2\rho)}{d_{\lambda_1}^2 d_{\lambda_2} d_{\lambda_3} 4(p+\alpha)^2} \\ &+ \frac{2[\beta+(1+\lambda)cd_1](c^3 + \{4(p-\alpha)^2 - c^2\}c)}{d_{\lambda_1} d_{\lambda_2}^2 2(p-\alpha)} + \frac{c^4 + 2c^2\{4(p-\alpha^2) - c^2\} + [\{4(p-\alpha)^2 - c^2\}]^2 2\rho}{d_{\lambda_2}^2 4(p-\alpha)} \end{aligned}\right\} (3.14)$$

And by elementary calculus $F'(\rho) > 0$ for $\rho > 0$. Thus implying that F is an increasing function and the upper bound for (3.13) correspond to $\rho = 1$ and $c = 0$ which implies (3.8).

Remark 3:
For $\alpha = \lambda = 0$ and $p = \beta = n = 1$ then

$$|a_2 a_4 - a_3^2| \le 1$$

which coincide with the result of Janteng et al. (2007).

Conclusion

The upper bounds for the functional $|a_{p+3}a_{p+1} - a_{p+2}^2|$ for functions belonging to $T_n^{\beta}(\alpha, p)$ and $T_n^{\beta}(\lambda, p)$ are obtained. And this coincide with earlier known results in univalent function theory.

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