REFINEMENTS OF THE EGYPTIAN FRACTION FINITE DIFFERENCE SCHEME FOR FIRST AND SECOND ORDER INITIAL VALUE PROBLEMS

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Abstract
In this paper, new methods, which are akin to both Runge-Kutta methods and Quasi-Runge-Kutta methods, through a refinement process by Taylor's series expansion of the error term of the existing Egyptian fraction method, were derived. They are less cumbersome than the Hybrid methods while maintaining high accuracy of the numerical results. The methods are used to solve both first and second order differential equations with initial conditions and the results obtained are very favourable because they produced lower absolute error when compared with the existing similar methods.

Keywords: *Refinement, Egyptian fraction, Runge-Kutta, Obrechkoff method*

Introduction
Numerical solution of Ordinary Differential Equation (ODE) is an important technique which has been developed over the years and many different methods are still being proposed and used for finding numerical approximations to the solutions of various types of ODE. However, there are a handful of methods known and used universally which can be categorized mainly as either one-step method or multistep method but the choice of an efficient method is dependent on two main criteria: speed and accuracy.

The Runge-Kutta methods are large class of one-step method algorithms with higher order of accuracy but have always been regarded as expensive because of their multiple function calls in each time step. For example, the four-stage Runge-Kutta method requires four evaluations of the function f per time step and the number of function evaluations is the method's computational Speed.

Another well known method is the block-hybrid method which has very high level of accuracy but it can be rigorous and therefore user's expertise is required.

The use of simple operations to find approximate solution (with high level of accuracy, speed and less complexity) to complex problems cannot be over emphasized in the field of numerical analysis. New methods are constantly being developed and existing methods being refined for the purpose of achieving this goal.

Egyptian Fraction
Ancient Egyptian hieroglyphics revealed to us much about the people of ancient Egypt, including how they did mathematics. The Rhind Mathematical Papyrus, the oldest existing mathematical manuscript, showed that their basic number system was very similar to ours except in one way – their concept of fractions. The ancient Egyptians had a way of writing numbers to at least one million. However, their method of writing fractions was limited. They had no other way of writing fractions, except for a special symbol for 2/3. This is not to say that the number 5/6 did not exist in ancient Egypt, they simply had no way of writing it as a single symbol. Instead, they would express it as a sum of unit fractions that is 1/2 + 1/3 (Izevhizua, 2008; Mohammad, 2012).

They only used fractions of the form $1/_n$ so that any other fraction had to be represented as a sum of such unit fractions and, furthermore, all the unit fractions were different. Their notation did not allow them to write $^3/_4$ or $^6/_7$ as we would today, instead, they were able to write any fraction as a sum of unit fractions.
For example;

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4} \qquad (2.1)$$

$$\frac{6}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42} \qquad (2.2)$$

where all the unit fractions were distinct. The unique fraction that the Egyptians did not represent using unit fractions was 2/3.

An infinite chain of unit fractions can be constructed using the identity

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} \qquad (2.3)$$

Therefore, a fraction written as a sum of distinct unit fractions is called an "Egyptian Fraction".

It is well-known that every positive rational number can be expressed as a sum of distinct unit fractions (reciprocals of natural numbers). Ancient Egyptians already used such representations of rational numbers, for this reason we call a sum of distinct unit fractions an Egyptian fraction. We note that sometimes unit fractions themselves are called Egyptian fractions.

Any rational number has representations as sum of Egyptian fractions with arbitrarily many terms and with arbitrarily large denominators, although for a given fixed number of terms, there are only finitely many Egyptian fractions.

Thus, *Egyptian fraction* is a term which now refers to any expression of a rational number as a sum of distinct unit fractions (a unit fraction is a reciprocal of a positive integer).

Refinement Process for the Egyptian Fraction Method
We consider the two-step Egyptian fraction scheme (Adeboye and Odio, 2013). given by:

$$y_{n+2} - y_n = \frac{h}{3}(f_{n+2} + 2f_{n+1} + 3f_n) \qquad (3.1)$$

For the solution of differential equation of the form $y' = f(x,y)$

The error term is expressed as:

$$y_{n+2} - y_n - \frac{h}{3}(f_{n+2} + 2f_{n+1} + 3f_n) \qquad (3.2)$$

For the refinement process (Salisu and Adeboye, 2012)., we expand (3.2) term by term in Taylor's series and substitute back, the error term is given as:

$$\left[y_n + 2hy'_n + \frac{2^2h^2}{2!}y''_n + \frac{2^3h^3}{3!}y'''_n + \dots\right] - [y_n]$$
$$- \frac{h}{3}\left[\left(y'_n + 2hy''_n + \frac{2^2h^2}{2!}y'''_n + \cdots\right) + 2\left(y'_n + hy''_n + \frac{h^2}{2!}y'''_n + \cdots\right) + 3y'_n\right]$$

Therefore, the error term is $\frac{2h^2}{3}y''_n + O(h^3)$ or $\frac{2h^2}{3}f'_n + O(h^3)$

We can express f'_n as $\left[\frac{f_{n+1} - f_n}{h}\right]$

That is;

$$\frac{2h^2}{3}\left[\frac{f_{n+1} - f_n}{h}\right] = \frac{2h}{3}[f_{n+1} - f_n] + O(h^3) \qquad (3.3)$$

Adding equation (3.3) to (3.1), we have

$$y_{n+2} - y_n = \frac{h}{3}(f_{n+2} + 2f_{n+1} + 3f_n) + \frac{2h}{3}[f_{n+1} - f_n] + O(h^3) \quad (3.4)$$

Thus, we have a new scheme 1 below:

Scheme1

$$y_{n+2} = y_n + \frac{h}{3}[f_{n+2} + 4f_{n+1} + f_n]$$

Also, we can express f'_n as $\left[\frac{f_{n+\frac{1}{4}} - f_n}{\frac{h}{4}}\right]$.

That is;

$$\frac{2h^2}{3}\left[\frac{f_{n+\frac{1}{4}} - f_n}{\frac{h}{4}}\right] = \frac{8h}{3}\left[f_{n+\frac{1}{4}} - f_n\right] + O(h^3) \quad (3.5)$$

Adding equation (3.5) to equation (3.1), we have

$$y_{n+2} - y_n = \frac{h}{3}(f_{n+2} + 2f_{n+1} + 3f_n) + \frac{8h}{3}\left[f_{n+\frac{1}{4}} - f_n\right] + O(h^3) \quad (3.6)$$

Thus, we have another new scheme 2 below:

Scheme 2

$$y_{n+2} = y_n + \frac{h}{3}\left[f_{n+2} + 2f_{n+1} + 8f_{n+\frac{1}{4}} - 5f_n\right]$$

Again, we can express f'_n as $\left[\frac{f_{n+1} - f_{n+\frac{1}{2}}}{\frac{h}{2}}\right]$.

That is:

$$\frac{2h^2}{3}\left[\frac{f_{n+1} - f_{n+\frac{1}{2}}}{\frac{h}{2}}\right] = \frac{4h}{3}\left[f_{n+1} - f_{n+\frac{1}{2}}\right] + O(h^3) \quad (3.7)$$

Adding equation (3.7) to equation (3.1), we have

$$y_{n+2} - y_n = \frac{h}{3}(f_{n+2} + 2f_{n+1} + 3f_n) + \frac{4h}{3}\left[f_{n+1} - f_{n+\frac{1}{2}}\right] + O(h^3) \quad (3.8)$$

Thus, we have the third scheme 3 below:

Scheme 3

$$y_{n+2} = y_n + \frac{h}{3}\left[f_{n+2} + 6f_{n+1} - 4f_{n+\frac{1}{2}} + 3f_n\right]$$

Convergence of the Methods

The necessary and sufficient conditions for a linear multistep method to be convergent are that it must be consistent and is zero-stable. [Lambert, 1974].

For Consistency:

Theorem 1: The linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j}$$

is said to be consistent with the differential equation:

$$y'(x) = f(x, y), \qquad y(x_0) = y_0$$

if and only if

i. $\rho(1) = 0$

ii. $\rho'(1) - \sigma(1) = 0$

where $\rho(\xi)$ and $\sigma(\xi)$ are the first and second characteristic polynomial

Zero-stability

Theorem 2: A linear multi-step method is zero-stable for any ordinary differential equation, if the roots of the first characteristic polynomial $\rho(\xi)$ lies inside the closed unit disc, with any root which lie on the unit circle being simple.

Scheme 1

$y_{n+2} - y_n = + \frac{h}{3}[f_{n+2} + 4f_{n+1} + f_n]$

For consistency:

$$\rho(\xi) = \sum_{j=0}^{2} \alpha_j \xi^j = \xi^2 - 1$$
$$\rho(1) = \sum_{j=0}^{2} \alpha_j = 1 - 1 = 0$$
$$\rho'(1) = \sum_{j=0}^{2} j\alpha_j = 2(1) - 0(1) = 2$$
$$\sigma(\xi) = \sum_{j=0}^{2} \beta_j \xi^j$$
$$\sigma(1) = \sum_{j=0}^{2} \beta_j = \frac{1}{3} + \frac{4}{3} - \frac{1}{3} = \frac{6}{3} = 2$$

For zero-stability:

$$\rho(\xi) = \xi^2 - 1 = (\xi + 1)(\xi - 1) = 0$$

$\xi = -1$ *and* 1 which satisfy the zero stability condition.

Since Scheme 1 is both consistent and zero-stable, it is therefore convergent

Scheme 2

$y_{n+2} - y_n = \frac{h}{3}\left(f_{n+2} + 2f_{n+1} + 8f_{n+\frac{1}{4}} - 5f_n\right)$

For consistency:

$$\rho(\xi) = \sum_{j=0}^{2} \alpha_j \xi^j = \xi^2 - 1$$
$$\rho(1) = \sum_{j=0}^{2} \alpha_j = 1 - 1 = 0$$
$$\rho'(1) = \sum_{j=0}^{2} j\alpha_j = 2(1) - 0(1) = 2$$
$$\sigma(\xi) = \sum_{j=0}^{2} \beta_j \xi^j$$
$$\sigma(1) = \sum_{j=0}^{2} \beta_j = \frac{1}{3} + \frac{2}{3} + \frac{8}{3} - \frac{5}{3} = \frac{6}{3} = 2$$

For zero-stability:

$$\rho(\xi) = \xi^2 - 1 = (\xi + 1)(\xi - 1) = 0$$

$\xi = -1$ *and* 1 which satisfy the zero stability condition.

Since Scheme 2 is both consistent and zero-stable, it is therefore convergent

Scheme 3

$y_{n+2} - y_n = \frac{h}{2}\left(f_{n+2} + 6f_{n+1} - 4f_{n+\frac{1}{2}} + 3f_n\right)$

For consistency:

$$\rho(\xi) = \sum_{j=0}^{2} \alpha_j \xi^j = \xi^2 - 1$$

$$\rho(1) = \sum_{j=0}^{2} \alpha_j = 1 - 1 = 0$$
$$\rho'(1) = \sum_{j=0}^{2} j\alpha_j = 2(1) - 0(1) = 2$$
$$\sigma(\xi) - \sum_{j=0}^{2} \beta_j \xi^j$$
$$\sigma(1) = \sum_{j-0}^{2} \beta_j = \frac{1}{3} + \frac{6}{3} - \frac{4}{3} + \frac{3}{3} = \frac{6}{3} = 2$$

For zero-stability:

$$\rho(\zeta) - \zeta^2 - 1 - (\zeta + 1)(\zeta - 1) - 0$$

$\xi = -1$ *and* 1 which satisfy the zero stability condition.
Since Scheme 3 is both consistent and zero-stable, it is therefore convergent

Numerical Experiments
Problem 1: $y' = f(x, y) = x + y; \quad y(x_0) = 1$
$y_E(x) = 2e^x - x - 1$
Problem 2: $y'' = y - 4x, \quad y(x_0) = 0, \; y'(x_0) = 1$
$y_E(x) = \frac{3}{2}(e^{-x} - e^x) + 4x$

Table 5.1: Solution to problem 1. Showing the absolute error of the new schemes compared with existing Adeboye's scheme at $h = 0.1$

x	Exact solution	Adeboye's Scheme	Scheme1	Scheme 2	Scheme 3
0.1	1.110341836	0.000000000	1.8448 x 10^{-4}	1.8448 x 10^{-4}	1.8448 x 10^{-4}
0.2	1.242805516	1.4506207 x 10^{-2}	2.57 x 10^{-5}	7.00252 x 10^{-4}	7.51654 x 10^{-4}
0.3	1.399717616	1.7034217 x 10^{-2}	2.01027 x 10^{-4}	6.250560 x 10^{-4}	1.127244 x 10^{-3}
0.4	1.583649396	3.5399604 x 10^{-2}	5.551 x 10^{-5}	1.715606 x 10^{-3}	1.851145 x 10^{-3}
0.5	1.797442542	4.1406275 x 10^{-2}	2.22891 x 10^{-4}	1.796314 x 10^{-3}	2.46557 x 10^{-3}
0.6	2.044237600	6.477844 x 10^{-2}	9.0462 x 10^{5}	3.151589 x 10^{-3}	3.412893 x 10^{-3}
0.7	2.327505414	7.5501445 x 10^{-2}	2.51162 x 10^{-4}	3.449316x 10^{-3}	4.331022 x 10^{-3}
0.8	2.651081856	1.0535194 x 10^{-1}	1.31807 x 10^{-4}	5.145035 x 10^{-3}	5.585484 x 10^{-3}
0.9	3.019206222	1.2239277 x 10^{-1}	2.87175 x 10^{-4}	5.739458 x 10^{-3}	6.893552 x 10^{-3}
1.0	3.436563656	1.6060829 x 10^{-1}	1.81074 x 10^{-4}	7.872758 x 10^{-3}	8.561106 x 10^{-3}

Table 5.2: Solution to problem 1, showing the absolute error of the new schemes compared with existing Adeboye's scheme at $h = 0.01$

x	Exact solution	Adeboye's Scheme	Scheme1	Scheme 2	Scheme 3
0.1	1.110341836	7.28223 x 10^{-4}	1.2x 10^{-8}	3.63 x 10^{-6}	3.665 x 10^{-6}
0.2	1.242805516	1.609525 x 10^{-3}	2.8 x 10^{-8}	8.024 x 10^{-6}	8.082 x 10^{-6}
0.3	1.399717616	2.668028 x 10^{-3}	4.1 x 10^{-8}	1.3301 x 10^{-5}	1.3402x 10^{-5}
0.4	1.583649396	3.931226 x 10^{-3}	5.6x 10^{-8}	1.9599 x 10^{-5}	1.9755 x 10^{-5}
0.5	1.797442542	5.430442 x 10^{-3}	7.3 x 10^{-8}	2.7078 x 10^{-5}	2.73x 10^{-5}
0.6	2.044237600	7.201318x 10^{-3}	9.3x 10^{-8}	3.5913 x 10^{-5}	3.6216 x 10^{-5}
0.7	2.327505414	9.284377 x 10^{-3}	1.12x 10^{-7}	4.6309 x 10^{-5}	4.6705 x 10^{-5}
0.8	2.651081856	1.1725658 x 10^{-2}	1.36 x 10^{-7}	5.8496 x 10^{-5}	5.9003x 10^{-5}
0.9	3.019206222	1.4577425x 10^{-2}	1.57 x 10^{-7}	7.2737 x 10^{-5}	7.3371 x 10^{-5}
1.0	3.436563656	1.7898966 x 10^{-2}	1.84x 10^{-7}	8.9324 x 10^{-5}	9.0119x 10^{-5}

Table 5.3: Solution of problem 2, showing absolute error of scheme 1 compared with Obrechkoff's method at $h = 0.1$

x	Exact solution	Obrechkoff's scheme	Scheme 1
0.1	0.0994997499	5.0083 x 10^{-6}	5.0041 x 10^{-6}
0.2	0.1959919924	1.00667 x 10^{-5}	1.00333 x 10^{-5}
0.3	0.2864391197	1.52258 x 10^{-5}	1.51124 x 10^{-5}
0.4	0.3677430226	2.05374 x 10^{-5}	2.02666 x 10^{-5}
0.5	0.4367140835	2.60545 x 10^{-5}	2.55207 x 10^{-5}
0.6	0.4900392536	3.18323 x 10^{-5}	3.08998 x 10^{-5}
0.7	0.5242488945	3.79287 x 10^{-5}	3.64286 x 10^{-5}
0.8	0.5356820534	4.44048 x 10^{-5}	4.21321 x 10^{-5}
0.9	0.5204498229	5.13252 x 10^{-5}	4.80349 x 10^{-5}
1.0	0.4743964191	5.87594 x 10^{-5}	5.41614 x 10^{-5}

Discussion of Results and Conclusion

In the first problem, the Egyptian fraction scheme together with its refined schemes are applied to first order Initial Valued Problem (IVP) and it can be seen in Table 5.1 and Table 5.2 that though the Egyptian fraction scheme is accurate, the new refined schemes are more accurate because they produce lower absolute error when compared with the original scheme and particularly, scheme 1 is with the lowest error.

Scheme 1 which is with the lowest error is further applied to second order IVP as demonstrated in Table 5.3 and Table 5.4 and is compared with the well-known Obrechkoff's method for second order problem. It can be seen that both methods have very high level of accuracy but scheme 1 is still with lower absolute error.

Table 5.4: Solution of problem 2, showing absolute error of scheme 1 compared with Obrechkoff's method at $h - 0.01$

x	Exact solution	Obrechkoff's scheme	Scheme 1
0.1	0.0994997499	5.002501 x 10^{-4}	5.002501 x 10^{-4}
0.2	0.1959919924	1.0055055 x 10^{-3}	1.0029993 x 10^{-3}
0.3	0.2864391197	1.5208219 x 10^{-3}	1.5107467 x 10^{-3}
0.4	0.3677430226	2.0513554 x 10^{-3}	2.0259909 x 10^{-3}
0.5	0.4367140835	2.6024144 x 10^{-3}	2.5512292 x 10^{-3}
0.6	0.4900392536	3.1795128 x 10^{-3}	3.0889558 x 10^{-3}
0.7	0.5242488945	3.7884248 x 10^{-3}	3.6416594 x 10^{-3}
0.8	0.5356820534	4.4352432 x 10^{-3}	4.2118188 x 10^{-3}
0.9	0.5204498229	5.1264399 x 10^{-3}	4.8018973 x 10^{-3}
1.0	0.4743964191	5.8689308 x 10^{-3}	5.4143349 x 10^{-3}

In conclusion, it can be observed that these are some Quasi-Runge-Kutta methods which are much simpler to obtain and implement than the Runge-Kutta methods and are also less cumbersome than the block methods but with comparable accuracy.

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