THE APPLICATION OF IMPROPER INTEGRAL IN EVALUATING ACTUARIAL RISK AND SINUSOIDAL TRANSFORM

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Abstract
Asymptotic domain analysis of certain improper integral could be performed by using information obtained from Gaussian distribution. Although the use of the Gaussian approach, a more specialized transform tool, eases out the computation acquisition techniques, the conversion becomes difficult. Not many analytic solutions exist for certain improper integral problems but in using the Gaussian function, we can deduce some of the closed form solution quite easily. The objective of the paper is to (i) develop a mortality model for an exponentially distributed actuarial mortality risk through the technique of factorial function and (ii) apply the Gaussian integral to evaluate improper integral involving sinusoidal transform. Integral transform finds application very much in actuarial domain and continuous time finance. The paper explores an analytical framework for evaluating the effect of structural properties of the Gaussian on certain improper integrals. In order to achieve this and create an analytically sound theoretical basis of evaluating improper integrals, the properties of the factorial and Gaussian functions are first examined. We then obtained the value of certain improper integrals among which are sine and complimentary sine transform. Specifically, for the presentation, we use the technique of Gaussian to derive asymptotic and approximation formulae in evaluating the integral of the form $\int_0^\infty x^{2n} e^{\frac{-x^2}{2}}\,dx$. *Our results reveal that (i)* $\int_0^\infty sin(xt)\, e^{\frac{-x^2}{2}}\,dx = e^{\frac{-t^2}{2}} \int_0^t e^{\frac{u^2}{2}}\,du + k$ *and (ii)* $\int_0^\infty cos(xt)\, e^{\frac{-x^2}{2}}\,dx = e^{-\frac{t^2}{2}+K_1}$ *and that for an exponentially distributed actuarial risk* T_x *(iii) the instantaneous force of intensity is approximately equal to the exponential parameter of the distribution.*

Keywords: factorial, improper integral, asymptotic, Gaussian function

Introduction
This Paper aims to (i) develop a mortality model for an exponentially distributed actuarial risk through the technique of factorial function defined by improper integral and (ii) apply the Gaussian integral to evaluate improper integral involving sinusoidal transform. Integral transform finds application very much in actuarial domain especially in mortality risk. The paper explores an analytical framework for evaluating the effect of structural properties of the Gaussian function on certain improper integrals of analysis and further calls the attention to some notable applications of Gaussian function in both actuarial probability and risk perspectives. Improper integral is employed as a basic tool to evaluate factorial function. Hamarsheh and Rajagopalan (2019), Mubeen and Rehman (2014), Kokologiannaki and Krasniqi (2013) and Mubeen and Habibullah (2012) gave various definitions of factorial function through the application of improper integral as a useful mathematical tool in some areas of statistics, actuarial mathematics and probability because it is capable of providing convenient results especially in Gamma distribution and related integrals

$$\int_0^\infty S^k e^{-s}\, dS = k! = k(k-1)(k-2)(k-3)(k-4)\dots 4.3.2.1$$
$$0! = 1$$
$$k! = k(k-1)!$$

Let g be a real valued function of a real variable.

$$g : Z^+ \cup \{-1, 0\} \rightarrow Z^+$$

$$g(k) = \begin{cases} 1 \text{ when } k = 0 \\ kg(k-1) \; k > 0, \quad k \in Z^+ \end{cases}$$
$$g(0) = 1$$
$$g(1) = 1 * g(0) = 1 * 1 = 1$$

$$g(2) = 2 * g(1) = 2$$

$$g(3) = 3 * g(2) = 6$$

$$g(4) = 4 * g(3) = 24$$
$$g(r) = r * g(r-1)$$

In, Mudunuru, *et al*. (2017), Mubeen and Rehman (2014), and Ibrahim (2013)

$$(k!)! = \begin{cases} k(k-2)(k-4)\dots 5.3.2.1 \;, k = 2m+1 \,, k = \text{odd} \\ k(k-2)(k-4)\dots 6.4.2.1, \; k = 2m, k = \text{even} \end{cases}$$
$$k! = k!!\,(k-1)!!$$

Note that $(k-1)$ can either be even or odd.

Material and Methods

Let X be a non-negative random variable suitable for representing random life time. Its cumulative distribution function $F_X(x) = \Pr[X \leq x]$ has the following four properties

$$\Pr(X = d) = \lim_{e \to d^-} \Pr[e < X \leq d] = F_X(d) - \lim_{e \to d^-} F_X(e)$$

$a < b$ implies $F_X(a) < F_X(b)$ = non – decreasing
$\lim_{d \to \infty} F_X(d) = 1$ = sum total of probabilities
$\lim_{d \to c^+} F_X(d) = F_X(c)$ = continuity from the right
$\lim_{d \to 0^-} F_X(d) = \mathbf{o}(1)$ = non- negativity

The probability density function of a random lifetime X be $f_X(x, \Theta)$ and the corresponding survival function be $S_X(x, \Theta)$, where Θ is the parameter of the density function and the distribution and survival functions are related by

$F_X(x, \Theta) = 1 - S_X(x, \Theta)$

By definition, in general, we might write $\int f_X(x, \Theta) = F_X(x, \Theta)$ where $f_X(x, \Theta)$ is a function of two variables. We dwell very much on common assumptions used in literature as stated below

Assumption: (i) $\frac{\partial f_X(x, \Theta)}{\partial \Theta}$ is continuous function in both x and Θ.

(ii) $\frac{\partial^2 F_X(x,\Theta)}{\partial \Theta\, \partial x} = \frac{\partial^2 F_X(x,\Theta)}{\partial x\, \partial \Theta}$

(iii) $f_X(x, \Theta)$ is continuous and non- negative in both x and Θ

$$\frac{\partial F_X(x,\ \Theta)}{\partial x} = f_X(x,\Theta) \tag{1}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F_X(x,\ \Theta)}{\partial \Theta}\right) = \frac{\partial}{\partial \Theta}\left(\frac{\partial F_X(x,\ \Theta)}{\partial x}\right) = \frac{\partial f_X(x,\ \Theta)}{\partial \Theta} \tag{2}$$

provided

$$\int \frac{\partial}{\partial x}\left(\frac{\partial F_X(x,\ \Theta)}{\partial \Theta}\right) d\Theta = \int \frac{\partial f_X(x,\ \Theta)}{\partial \Theta}\, d\Theta \tag{3}$$

$$\int \frac{\partial}{\partial x}\left(\frac{\partial F_X(x,\ \Theta)}{\partial \Theta}\right) dx = \int \frac{\partial f_X(x,\ \Theta)}{\partial \Theta}\, dx \tag{4}$$

$$\int \frac{\partial}{\partial x}\left(\frac{\partial F_X(x,\ \Theta)}{\partial \Theta}\right) dx = \frac{\partial}{\partial \Theta}\int f_X(x,\ \Theta)\, dx = \frac{\partial}{\partial \Theta} F_X(x,\Theta) \tag{5}$$

We define $J(\Theta) = \int_a^b f_X(x,\Theta)\, dx$ (6)

$f_X(x,\Theta)$ is integrable function of X in the $x \in [a\ b]$

The probability that a newborn dies between $X = a(\Theta)$ and $X = b(\Theta)$ is

$$\Pr(a \le X \le b) = \int_a^b f_X(x,\Theta)\, dx = F_X(b,\Theta) - F_X(a,\Theta) \tag{7}$$

$$\Pr(a \le X \le \omega) = \int_a^\omega f_X(x,\Theta)\, dx = F_X(\omega,\Theta) - F_X(a,\Theta) \tag{8}$$

where ω is the terminal age beyond which no life exist. If $a = 0$,

then $\Pr(0 \le X \le \omega) = \int_0^\omega f_X(x,\Theta)\, dx = 1$ (9)

The probability that a newborn dies between $X = x$ and $X = x + \delta x$ is

$$\Pr(x \le X \le x + \delta x) = \int_x^{x+\delta x} f_X(x,\Theta)\, dx \tag{10}$$

$$\Pr(x \le X \le x + \delta x) = F_X(x + \delta x,\ \Theta) - F_X(x,\Theta) \tag{11}$$

$$J(\Theta) = \int_a^b f_X(x,\Theta)\, dx = F_X(b,\Theta) - F_X(a,\Theta) \tag{12}$$

$$\frac{\partial}{\partial \Theta}\int_a^b f_X(x,\Theta)\, dx = \frac{\partial F_X(b,\ \Theta)}{\partial \Theta} - \frac{\partial F_X(a,\ \Theta)}{\partial \Theta} \tag{13}$$

taking total derivatives, we have

$$dF_X(b,\Theta) = \frac{\partial F_X(b,\Theta)}{\partial \Theta} d\Theta + \frac{\partial F_X(b,\Theta)}{\partial b} db \tag{14}$$

$$\frac{dF_X(b,\Theta)}{\partial \Theta} = \frac{\partial F_X(b,\Theta)}{\partial \Theta} + \frac{\partial F_X(b,\Theta)}{\partial b}\frac{db}{d\Theta} \tag{15}$$

similarly

$$\frac{dF_X(a,\Theta)}{\partial \Theta} = \frac{\partial F_X(a,\Theta)}{\partial \Theta} + \frac{\partial F_X(a,\Theta)}{\partial a}\frac{da}{d\Theta} \tag{16}$$

$$\frac{dJ(\Theta)}{d\Theta} = \frac{\partial F_X(b,\Theta)}{\partial \Theta} + \frac{\partial F_X(b,\Theta)}{\partial b}\frac{db}{d\Theta} - \left\{\frac{\partial F_X(a,\Theta)}{\partial \Theta} + \frac{\partial F_X(a,\Theta)}{\partial a}\frac{da}{d\Theta}\right\} \tag{17}$$

$$\frac{dJ(\Theta)}{d\Theta} = \frac{\partial F_X(b,\Theta)}{\partial \Theta} + \frac{\partial F_X(b,\Theta)}{\partial b}\frac{db}{d\Theta} - \frac{\partial F_X(a,\Theta)}{\partial \Theta} - \frac{\partial F_X(a,\Theta)}{\partial b}\frac{da}{d\Theta} \tag{18}$$

$$\frac{dJ(\Theta)}{d\Theta} = \frac{\partial F_X(b,\Theta)}{\partial \Theta} - \frac{\partial F_X(a,\Theta)}{\partial \Theta} + \frac{\partial F_X(b,\Theta)}{\partial b}\frac{db}{d\Theta} - \frac{\partial F_X(a,\Theta)}{\partial b}\frac{da}{d\Theta} \tag{19}$$

$$\frac{dJ(\Theta)}{d\Theta} = \int_a^b f_X(x,\Theta)\, dx + f_X(b,\Theta)\frac{db}{d\Theta} - f_X(a,\Theta)\frac{da}{d\Theta} \tag{20}$$

Calculus of Factorial Function

Young (2017) defines $n! = \int_0^\infty x^n e^{-x}\,dx$ (21)

$0! = \int_0^\infty x^0 e^{-x}\,dx = \int_0^\infty e^{-x}\,dx$ (22)

Letting $x = tu$. Then $dx = tdu$ and

$\int_0^\infty te^{-tu}\,du = 1$ (23)

$\int_0^\infty e^{-tx}\,dx = \frac{1}{t}$ (24)

Differentiating both sides of (24) with respect to t, that is

$\frac{d}{dt}\int_0^\infty e^{-tx}\,dx = \frac{d}{dt}\left(\frac{1}{t}\right)$ (25)

$\int_0^\infty -xe^{-tx}\,dx = \frac{-1}{t^2}$ (26)

hence $\int_0^\infty xe^{-tx}\,dx = \frac{\int_0^\infty x^1 e^{-x}dx}{t^2}$ (27)

Differentiating both sides of (26) with respect to t again

$\int_0^\infty -x^2 e^{-tx}\,dx = \frac{-2}{t^3}$ hence $\int_0^\infty x^2 e^{-tx}\,dx = \frac{\int_0^\infty x^2 e^{-x}dx}{t^3}$ (28)

$\int_0^\infty x^3 e^{-tx}\,dx = \frac{\int_0^\infty x^3 e^{-x}dx}{t^4}$ (29)

$\int_0^\infty x^4 e^{-tx}\,dx = \frac{\int_0^\infty x^4 e^{-x}dx}{t^5}$ (30)

$\int_0^\infty x^5 e^{-tx}\,dx = \frac{\int_0^\infty x^5 e^{-x}dx}{t^6}$ (31)

$\int_0^\infty x^6 e^{-tx}\,dx = \frac{\int_0^\infty x^6 e^{-x}dx}{t^7}$ (32)

$\int_0^\infty x^n e^{-tx}\,dx = \frac{n!}{t^{n+1}} = \frac{\int_0^\infty x^n e^{-x}dx}{t^{n+1}}$ (33)

When $t = 1$

$\int_0^\infty x^n e^{-x}\,dx = n!$ (34)

In order to test the principles developed above, the exponential distribution is examined to this effect.

Theorem 2.1: Let the random lifetime of a new born X be exponentially distributed with parameter Θ. Then $\mu(x) = \Theta$ (35)

Proof

$f_X(x, \Theta) = \Theta e^{-\Theta x}$ (36)

$\int_0^\infty \Theta e^{-\Theta x}dx = 1$ (37)

$\int_0^\infty e^{-\Theta x}dx = \frac{1}{\Theta}$ (38)

$\frac{d^n[\int_0^\infty e^{-\Theta x}dx]}{d\Theta^n} = \frac{d^n}{d\Theta^n}\{\frac{1}{\Theta}\},\ n \geq 0$ (39)

$\int_0^\infty \frac{d^n}{d\Theta^n} e^{-\Theta x}\,dx = \frac{d^n}{d\Theta^n}\{\frac{1}{\Theta}\}$ (40)

$\int_0^\infty (-1)^n x^n e^{-\Theta x}\,dx = \frac{(-1)^n\, n!}{\Theta^{n+1}}$ (41)

$\int_0^\infty x^n e^{-\Theta x}\,dx = \frac{n!}{\Theta^{n+1}}$ (42)

When $n = 0$, $\int_0^\infty e^{-\Theta x}\,dx = \frac{1}{\Theta}$ (43)

If $\Theta = \frac{1}{\alpha}$, $\int_0^\infty e^{-\frac{1}{\alpha}x}\,dx = \alpha$ (44)

When $n = 1$, $\int_0^\infty x^1 e^{-\Theta x}\,dx = \frac{1}{\Theta^2}$ (45)

$$\langle X\rangle = \int_0^\infty \Theta x^1 e^{-\Theta x}\, dx = \frac{1}{\Theta} \tag{46}$$

When $n = 2$, $\int_0^\infty x^2 e^{-\Theta x}\, dx = \frac{2}{\Theta^3}$ (47)

$$\langle X^2\rangle = \int_0^\infty \Theta x^2 e^{-\Theta x}\, dx = \frac{2}{\Theta^2} \tag{48}$$

$$\sigma^2{}_X = \frac{2}{\Theta^2} - \frac{1}{\Theta^4} = \frac{2\Theta^2 - 1}{\Theta^4} \tag{49}$$

When $n = 3$, $\int_0^\infty x^3 e^{-\Theta x}\, dx = \frac{6}{\Theta^4}$ (50)

$$F_X(x,\Theta) = \int_0^x \Theta e^{-\Theta u} du = -e^{-\Theta u}\Big|_0^x = 1 - e^{-\Theta x} \tag{51}$$

$$S_X(x,\Theta) = e^{-\Theta x},\ l_X = l_0 S_X(x,\Theta) \tag{52}$$

$$l_X = l_0 e^{-\Theta x} \tag{53}$$

$$l_x{}' = -\Theta l_0 e^{-\Theta x} \tag{54}$$

$$\mu_X(x) = \frac{-l_x{}'}{l_X} = \Theta \tag{55}$$

Following, Ogungbenle and Adeyele (2020), the instantaneous rate of mortality is estimated as

$$\Theta = \left\{\frac{3\,l_{x+4} + 36\,l_{x+2} + 25\,l_x - 16\,l_{x+3} - 48\,l_{x+1}}{12\,l_x}\right\} \tag{56}$$

Where l_x is the expected number of survivors at age x, l_0 is the radix of mortality table and Θ is the estimated mortality intensity.

Methods and Solutions

Euler's Integral and Normal Distribution

Among all the continuous probability distributions, the normal distribution is very prominent because it evolves in many applications. As we see in Straub (2014), Chesneau and Navarro (2018), Chesneau and Navarro (2019), its main use lies in the central limit theorem where the sample mean has a normal distribution if the sample size is very large. In view of Soranzo and Epure (2014), using the normal approximation, we estimate the Euler's integral with the Gaussian distribution using linear transformation $\sigma y = x - \mu$. This approximation can be justified by the central limit theorem, since the sum of independent random variables tends to a normal random variable as the number in the sum increases. A random risk X is said to have a normal distribution if its probability density function is given by. In order to ease out the evaluation of sinusoidal transform, we first of all examine the Gaussian function below

$$f(x) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{1}{2}\left[\frac{X-\mu}{\sigma}\right]^2},\ -\infty < x < \infty \tag{57}$$

let, $y = \frac{X-\mu}{\sigma} \Rightarrow \sigma y = x - \mu$ (58)

$$\sigma dy = dx \tag{59}$$

$$\phi(x) = \frac{1}{\sigma\sqrt{2\Pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{X-\mu}{\sigma}\right]^2} dx = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[y]^2} dy = \frac{1}{\sigma} \tag{60}$$

$$g(y) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}[y]^2} dy = \sqrt{2\Pi} \tag{61}$$

Normal distribution was originally developed as an approximation to the binomial distribution when the number of trials is large and the Bernoulli probability p is not close to 0 or 1. It is also the asymptotic form of the sum of random variables under a wide range of conditions. In Kallenberg (1997), we observe that the development of the distribution is often ascribed to Gauss who applied the theory to the movements of celestial bodies. The key reason is that large sums of small random variables often turn out to be normally distributed. $Z \sim N(0,1)$ is normally distributed with parameters 0 and 1. The distribution function is denoted

$$\phi(x) = \frac{1}{\sigma\sqrt{2\Pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}[y]^2}\, dy \tag{62}$$

Let $I(t) = \int_0^{\infty} \frac{e^{\frac{-(1+x^2)t^2}{2}}}{1+x^2}\, dx$ (63)

Letting $x = \tan A$

$$I(0) = \int_0^{\infty} \frac{1}{1+x^2}\, dx = \frac{\Pi}{2} \quad \text{i.e, when } x = \mu \tag{63b}$$

$$\frac{\partial I(t)}{\partial t} = \int_0^{\infty} -t\, e^{\frac{-(1+x^2)t^2}{2}}\, dx \tag{64}$$

$$\frac{\partial I(t)}{\partial t} = -t\, e^{\frac{-t^2}{2}} \int_0^{\infty} e^{\frac{-(tx)^2}{2}}\, dx \tag{65}$$

Let $y = tx$, then $dy = tdx$

$$\frac{\partial I(t)}{\partial t} = -e^{\frac{-t^2}{2}} \int_0^{\infty} e^{\frac{-y^2}{2}}\, dy = -e^{\frac{-t^2}{2}} \int_0^{\infty} e^{\frac{-x^2}{2}}\, dx \tag{66}$$

$$\frac{\partial I(t)}{\partial t} = \int_0^{\infty} e^{\frac{-x^2}{2}}\, dx \quad \int_0^{k} e^{\frac{-t^2}{2}}\, dt, k \to \infty \tag{67}$$

$$0 - \frac{\Pi}{2} = -[\int_0^{\infty} e^{\frac{-x^2}{2}}\, dx]^2 \tag{68}$$

$$\int_0^{\infty} e^{\frac{-x^2}{2}}\, dx = \sqrt{\frac{\Pi}{2}} = \frac{\sqrt{2\Pi}}{2} \tag{69}$$

$$\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}\, dx = \sqrt{2\Pi} \tag{70}$$

$$n = 0, \quad \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}}\, dy = \sqrt{2\Pi} \tag{71}$$

Let $y = x\sqrt{t} \Rightarrow dy = \sqrt{t}\, dx$ (72)

$$\int_{-\infty}^{\infty} e^{\frac{-tx^2}{2}} \sqrt{t} dx = \sqrt{2\Pi} \tag{73}$$

$$\int_{-\infty}^{\infty} e^{\frac{-tx^2}{2}}\, dx = \sqrt{2\Pi}\, t^{-\frac{1}{2}} \tag{74}$$

Differentiate both sides of (74) with respect to t, using differentiation under the integral sign on the left,

$$\int_{-\infty}^{\infty} \frac{-x^2}{2} e^{\frac{-tx^2}{2}}\, dx = -\frac{1}{2}\sqrt{2\Pi}\, t^{-\frac{3}{2}} \Rightarrow \tag{75}$$

$$\int_{-\infty}^{\infty} x^2 e^{\frac{-tx^2}{2}}\, dx = \sqrt{2\Pi}\, t^{-\frac{3}{2}} \tag{76}$$

Differentiate both sides of (75) with respect to t, using differentiation under the integral sign on the left and so on like that, we have

$$\int_{-\infty}^{\infty} \frac{x^4}{4} e^{\frac{-tx^2}{2}}\, dx = \frac{3}{4}\sqrt{2\Pi}\, t^{-\frac{5}{2}} \Rightarrow \int_{-\infty}^{\infty} x^4 e^{\frac{-tx^2}{2}}\, dx = 1 * 3\sqrt{2\Pi}\, t^{-\frac{5}{2}} \tag{77}$$

$$\int_{-\infty}^{\infty} \frac{x^6}{8} e^{\frac{-tx^2}{2}}\, dx = \frac{15}{8}\sqrt{2\Pi}\, t^{-\frac{7}{2}} \Rightarrow \int_{-\infty}^{\infty} x^6 e^{\frac{-tx^2}{2}}\, dx = 1 * 3 * 5\sqrt{2\Pi}\, t^{-\frac{7}{2}} \tag{78}$$

$$\int_{-\infty}^{\infty} \frac{x^8}{16} e^{\frac{-tx^2}{2}}\, dx = \frac{15x7}{16}\sqrt{2\Pi}\, t^{-\frac{9}{2}} \Rightarrow \int_{-\infty}^{\infty} x^8 e^{\frac{-tx^2}{2}}\, dx = 1 * 3 * 5 * 7\sqrt{2\Pi}\, t^{-\frac{9}{2}} \tag{79}$$

$$\int_{-\infty}^{\infty} \frac{x^{10}}{32} e^{\frac{-tx^2}{2}}\, dx = \frac{15x7x\,9}{32}\sqrt{2\Pi}\, t^{-\frac{11}{2}} \tag{80}$$

$$\int_{-\infty}^{\infty} x^{10} e^{\frac{-tx^2}{2}}\, dx = 1 * 3 * 5 * 7 * 9\sqrt{2\Pi}\, t^{-\frac{11}{2}}$$

$$\int_{-\infty}^{\infty} x^m e^{\frac{-tx^2}{2}} dx = 1*3*5*\ldots*X\,(m-1)\sqrt{2\Pi}\, t^{-\frac{m-1}{2}} \quad (81)$$

$$\int_{-\infty}^{\infty} x^m e^{\frac{-tx^2}{2}} dx = 1*3*5*\ldots*(m-1)\, t^{-\frac{m}{2}} \sqrt{\frac{2\Pi}{t}} \quad (82)$$

When $t \longrightarrow 1$

$$\int_{-\infty}^{\infty} x^m e^{\frac{-x^2}{2}} dx = 1*3*5*\ldots*(m-1)\sqrt{2\Pi} \quad (83)$$

When $t \longrightarrow 2$

$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = 1*3*5*\ldots*(m-1)\, 2^{-\frac{m}{2}}\sqrt{\Pi}$ using (63b), we have

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = 1*3*5*\ldots*(m-1)\, 2^{-\frac{m}{2}}\sqrt{2I(0)}$$

Evaluation of the Sinusoidal Sine Transform Using the Gaussian Distribution

In view of Yip (2000), Young (2017) and Zhang and Zhang (2019), integral transforms using sine and cosine functions as the integral kernels specify a grey area in mathematical analysis. It is dependent on the half range expansion of function over a set of cosine or sine basis functions. Since cosine and sine kernels do not possess the nice properties structure of an exponential kernel, the transform properties do not seem to be elegant but much more involving than the corresponding ones for the Fourier transform kernel. In view of these noted computational limitations, sine and cosine transforms have useful applications especially in analysis and modeling of seasonal interest rates, present value analysis, spectral analysis of real sequences, in solutions of some boundary value problems and in transform domain processing of digital signals,

$$S(t) = \int_0^{\infty} \sin(xt)\, e^{\frac{-x^2}{2}} dx, \quad S(0) = 0 \quad (84)$$

$$S'(t) = \int_0^{\infty} x\cos(xt)\, e^{\frac{-x^2}{2}} dx, \quad S'(0) = \int_{-\infty}^{\infty} xe^{\frac{-x^2}{2}} dx \quad (85)$$

$$S''(t) = -\int_0^{\infty} x^2\sin(xt)\, e^{\frac{-x^2}{2}} dx, S''(0) = 0 \quad (86)$$

$$S'''(t) = -\int_0^{\infty} x^3\cos(xt)\, e^{\frac{-x^2}{2}} dx, \quad S'''(0) = -\int_0^{\infty} x^3 e^{\frac{-x^2}{2}} dx \quad (87)$$

$$S^{(iv)}(t) = \int_0^{\infty} x^4\sin(xt)\, e^{\frac{-x^2}{2}} dx, \quad S^{(4)}(0) = 0 \quad (88)$$

$$S^{(2m)}(t) = 0, \quad S^{(2m+1)}(t) = (-1)^{m+2}\int_0^{\infty} \mathbf{x^{2m+1}e^{\frac{-x^2}{2}}} dx, m = 0,1,2,3,\ldots \quad (89)$$

$$S^{(2m+1)}(t) = (-1)^{m+2} x^{m+1}\int_0^{\infty} x^m e^{\frac{-x^2}{2}} dx \quad (90)$$

$$\frac{S^{(2m+1)}(t)}{(-1)^{m+2}x^{m+1}} = \frac{[1*3*5*7*9\ldots(m-1)\sqrt{2\Pi}\,]}{2} \quad (91)$$

This is the relation between the sine transform and Euler's integral.

Theorem 3.2

$$\int_0^{\infty} \sin(xt)\, e^{\frac{-x^2}{2}} dx \sim \frac{1}{t}$$

Proof

$$S'(t) = \int_0^{\infty} x\cos(xt)\, e^{\frac{-x^2}{2}} dx \quad (92)$$

we integrate by parts to obtain

$$u = \cos(xt), dv = x\, e^{\frac{-x^2}{2}} \quad (93)$$

$$du = -t\sin(xt)dx, v = \int x\, e^{\frac{-x^2}{2}} dx \quad (94)$$

letting $U = x^2 \Rightarrow dU = 2xdx$ so that

$$v = \int x\, e^{\frac{-U}{2}} \frac{dU}{2x} = \frac{1}{2} \int e^{\frac{-U}{2}} du = -e^{\frac{-U}{2}} \quad (95)$$

$$v = -e^{\frac{-x^2}{2}} \quad (96)$$

$$S'(t) = -e^{\frac{-x^2}{2}} \cos(xt) - t \int_0^\infty e^{\frac{-x^2}{2}} \sin(xt)\, dx \quad (97)$$

$$S'(t) = -e^{\frac{-x^2}{2}} \cos(xt) \Big|_0^\infty - t\, S(t) \quad (98)$$

$$S'(t) = -[0-1] - t\, S(t) \quad (99)$$

$$S'(t) = 1 - t\, S(t) \quad (100)$$

$$S'(t) + t\, S(t) - 1 = 0 \quad (101)$$

the integrating factor is $\mu(t) = e^{\int_0^t u du} = e^{\frac{t^2}{2}}$ (102)

$$S'(t) e^{\frac{t^2}{2}} + t\, S(t) e^{\frac{t^2}{2}} - e^{\frac{t^2}{2}} = 0 \quad (103)$$

$$\left(S(t) e^{\frac{t^2}{2}} \right)' = e^{\frac{t^2}{2}} \quad (104)$$

$$S(t) e^{\frac{t^2}{2}} = \int_0^t e^{\frac{u^2}{2}} du + k \quad (105)$$

$$S(t) = e^{\frac{-t^2}{2}} \int_0^t e^{\frac{u^2}{2}} du + k$$

$$S(0) = k = 0 \quad (106)$$

$$\int_0^\infty \sin(xt)\, e^{\frac{-x^2}{2}} dx = e^{\frac{-t^2}{2}} \int_0^t e^{\frac{u^2}{2}} du + 0 \quad (107)$$

$$\left(when\ t = 1, \int_0^\infty \sin(x)\, e^{\frac{-x^2}{2}} dx = e^{\frac{-1}{2}} \int_0^1 e^{\frac{u^2}{2}} du \right) \quad (107b)$$

To compute the integral at the right hand side, we split the range of the integral arbitrarily

$$\int_0^t e^{\frac{u^2}{2}} du = \int_0^1 e^{\frac{u^2}{2}} du + \int_1^t \frac{1}{u} \frac{de^{\frac{u^2}{2}}}{du} du$$

$$\int_0^t e^{\frac{u^2}{2}} du = \int_0^1 e^{\frac{u^2}{2}} du + \left(\left[\frac{e^{\frac{u^2}{2}}}{u} \right] \Bigg|_{u=1}^{u=t} \right) + \int_0^t \frac{e^{\frac{u^2}{2}}}{u} du$$

$$\int_0^t e^{\frac{u^2}{2}} du = \int_0^1 e^{\frac{u^2}{2}} du + \frac{1}{t} e^{\frac{t^2}{2}} - e^{\frac{1}{2}} + \int_0^t \frac{1}{u} e^{\frac{u^2}{2}} du$$

From my observation, the expression on the right hand side is dominated by $\frac{1}{t} e^{\frac{t^2}{2}}$ to the extent that as $t \longrightarrow \infty$, each term at the right hand side is less than $\frac{1}{t} e^{\frac{t^2}{2}}$

$$\int_0^t e^{\frac{u^2}{2}} du \sim \frac{1}{t} e^{\frac{t^2}{2}}$$

$$\int_0^\infty \sin(xt)\, e^{\frac{-x^2}{2}} dx \sim \frac{1}{t}$$

The Complementary Sine Transform of the Gaussian Distribution

We observe in Polyanin & Manzhirov (2008); Yip (2000); Young (2017), and Zhang and Zhang (2019) that the complementary sine integral transform is defined as

$$C(t) = \int_0^\infty \cos(xt)\, e^{\frac{-x^2}{2}} dx \quad (108)$$

This is equivalent to replacing $\sin(xt)$ in equation (84) by $\cos(xt)$

$$C(0) = \int_0^\infty \cos(0)\, e^{\frac{-x^2}{2}}\, dx = \int_0^\infty e^{\frac{-x^2}{2}}\, dx = \sqrt{\frac{\Pi}{2}} \tag{109}$$

$$C'(t) = \int_0^\infty -x^1 \sin(xt)\, e^{\frac{-x^2}{2}}\, dx \tag{110}$$

$$C'(0) = \int_0^\infty -0 * \sin(0)\, e^{\frac{-x^2}{2}}\, dx = 0 \tag{111}$$

$$C''(t) = \int_0^\infty -x^2 \cos(xt)\, e^{\frac{-x^2}{2}}\, dx \tag{112}$$

$$C''(0) = \int_0^\infty -x^2 \cos(0)\, e^{\frac{-x^2}{2}}\, dx = -\int_0^\infty x^2 e^{\frac{-x^2}{2}}\, dx \tag{113}$$

$$C'''(t) = \int_0^\infty x^3 \sin(xt)\, e^{\frac{-x^2}{2}}\, dx \tag{114}$$

$$C'''(0) = 0$$

$$C^{(iv)}(t) = \int_0^\infty x^4 \cos(xt)\, e^{\frac{-x^2}{2}}\, dx \tag{115}$$

$$C^{(iv)}(0) = \int_0^\infty x^4 \cos(0)\, e^{\frac{-x^2}{2}}\, dx = \int_0^\infty x^4 e^{\frac{-x^2}{2}}\, dx \tag{116}$$

$$C^{(v)}(t) = \int_0^\infty -x^5 \sin(xt)\, e^{\frac{-x^2}{2}}\, dx \tag{117}$$

$$C^{(v)}(0) = \int_0^\infty -x^5 \sin(0)\, e^{\frac{-x^2}{2}}\, dx = 0$$

$$C^{(vi)}(t) = \int_0^\infty -x^6 \cos(xt)\, e^{\frac{-x^2}{2}}\, dx \tag{118}$$

$$C^{(vi)}(0) = \int_0^\infty -x^6 e^{\frac{-x^2}{2}}\, dx \tag{119}$$

$$C^{(2k)}(t) = (-1)^n \int_0^\infty x^{2k} \cos(xt)\, e^{\frac{-x^2}{2}}\, dx \tag{120}$$

$$C^{(2k)}(0) = (-1)^n \int_0^\infty x^{2k} e^{\frac{-x^2}{2}}\, dx,\ k = 0, 1, 2, 3, 4, \dots n \tag{121}$$

Theorem 3.3

$$\int_0^\infty \cos(xt)\, e^{\frac{-x^2}{2}}\, dx = e^{-\frac{t^2}{2}+K_1} \tag{122}$$

Proof

$$C'(t) = \int_0^\infty -x^1 \sin(xt)\, e^{\frac{-x^2}{2}}\, dx \tag{123}$$

$$u = \sin(xt), dv = x\, e^{\frac{-x^2}{2}} \tag{124}$$

$$du = t\cos(xt)dx,\ v = \int x\, e^{\frac{-x^2}{2}}\, dx \tag{125}$$

letting $U = x^2 \Rightarrow dU = 2xdx$ so that

$$v = \int x\, e^{\frac{-U}{2}} \frac{dU}{2x} = \frac{1}{2} \int e^{\frac{-U}{2}}\, du = -\, e^{\frac{-U}{2}} \tag{126}$$

$$v = -\, e^{\frac{-x^2}{2}} \tag{127}$$

$$-C'(t) = -\, e^{\frac{-x^2}{2}} \sin(xt) - t \int_0^\infty -e^{\frac{-x^2}{2}} \cos(xt)\, dx \tag{128}$$

$$C'(t) = e^{\frac{-x^2}{2}} \sin(xt) - t \int_0^\infty e^{\frac{-x^2}{2}} \cos(xt)\, dx \tag{129}$$

$$C'(t) = e^{\frac{-x^2}{2}} \sin(xt) \Big|_0^\infty - t\, C(t) \tag{130}$$

$$C'(t) = e^{\frac{-x^2}{2}} \sin(xt) \Big|_0^\infty - t\, C(t) \tag{131}$$

$$C'(t) = -t\ C(t) \tag{132}$$

$$\frac{C'(t)}{C(t)} = -t \Rightarrow dlog_e C(t) = -t \tag{133}$$

$$log_e C(t) = \frac{-t^2}{2} + K_1 \tag{134}$$

$$C(t) = e^{-\frac{t^2}{2}+K_1} \tag{135}$$

$$C(0) = e^{\frac{2K_1}{2}} = \sqrt{\frac{\Pi}{2}} \tag{136}$$

$$K_1 = \log_e \left[\sqrt{\frac{\Pi}{2}}\right] \tag{137}$$

$$\int_0^\infty \cos(xt)\, e^{\frac{-x^2}{2}}\, dx = \log_e \left[\sqrt{\frac{\Pi}{2}}\right] e^{-\frac{t^2}{2}} \tag{138}$$

$$S(t) + C(t) = e^{\frac{-t^2}{2}} \left[\int_0^t e^{\frac{u^2}{2}}\, du + \log_e \left[\sqrt{\frac{\Pi}{2}}\right]\right] \tag{139}$$

$$S(0) + C(0) = \left[\log_e \left[\sqrt{\frac{\Pi}{2}}\right]\right] \text{ and}$$

$$S(1) + C(1) = e^{\frac{-1}{2}} \left[\int_0^1 e^{\frac{u^2}{2}}\, du + \log_e \left[\sqrt{\frac{\Pi}{2}}\right]\right]$$

Discussion of Results

In this paper we have presented different properties of the Gaussian function as applicable in sinusoidal problems, provided with simple proofs. However, based on the definitions in section 3, we present the following discussions

$$C(t) = \int_0^\infty \cos(xt)\, e^{\frac{-x^2}{2}}\, dx = e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right] - t^2}{2}} \tag{140}$$

$$C(0) = \int_0^\infty e^{\frac{-x^2}{2}}\, dx = e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]}{2}} \tag{141}$$

$$C'(t) = -2t\, e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right] - t^2}{2}} \tag{142}$$

$$C'(t) = \int_0^\infty -x^1 \sin(xt)\, e^{\frac{-x^2}{2}}\, dx = -2t\, e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right] - t^2}{2}} \tag{143}$$

$$C'(0) = 0$$

$$C''(t) = -2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right] - t^2}{2}} + 4t^2\, e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right] - t^2}{2}} \tag{144}$$

$$C''(t) = \int_0^\infty -x^2 \cos(xt)\, e^{\frac{-x^2}{2}}\, dx = -2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right] - t^2}{2}} + 4t^2\, e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right] - t^2}{2}} \tag{145}$$

$$C''(0) = \int_0^\infty -x^2 e^{\frac{-x^2}{2}}\, dx = -2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]}{2}} \tag{146}$$

$$C'''(t) = 4te^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 8t\,e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 8t^3e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (147)$$

$$C'''(t) = 12te^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 8t^3e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (148)$$

$$C'''(t) = \int_0^\infty x^3\sin(xt)\,e^{\frac{-x^2}{2}}dx = 12te^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 8t^3e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (149)$$

$$C'''(0) = 0$$

$$C^{iv}(t) = 12e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 24t^2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 24t^2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 16t^4e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (150)$$

$$C^{iv}(t) = 12e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 48t^2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 16t^4e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (151)$$

$$\int_0^\infty x^4\cos(xt)\,e^{\frac{-x^2}{2}}dx = 12e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 48t^2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 16t^4e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (152)$$

$$C^{iv}(0) = \int_0^\infty x^4e^{\frac{-x^2}{2}}dx = 12e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]}{2}} \quad (153)$$

$$C^{v}(t) = -24te^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 96t\,e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 96t^3e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 64t^3e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 32t^5e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (154)$$

$$C^{v}(t) = -120te^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 32t^3e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 32t^5e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (155)$$

$$\int_0^\infty -x^5\sin(xt)\,e^{\frac{-x^2}{2}}dx = -120te^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 32t^3e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 32t^5e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (156)$$

$$C^{(vi)}(t) = -120e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 240t^2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 96t^2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 64t^4e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 160t^4e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 64t^6e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (157)$$

$$C^{(vi)}(t) = -120e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 144t^2e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 96t^4e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 64t^6e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}}$$

$$\int_0^\infty -x^6\cos(xt)\, e^{\frac{-x^2}{2}}\, dx = -120e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 144t^2 e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} - 96t^4 e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} + 64t^6 e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]-t^2}{2}} \quad (158)$$

$$C^{(vi)}(0) = \int_0^\infty -x^6 e^{\frac{-x^2}{2}}\, dx = -120e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]}{2}} \quad (159)$$

implying

$$\int_0^\infty x^6 e^{\frac{-x^2}{2}}\, dx = 120e^{\frac{2\log_e\left[\sqrt{\frac{\Pi}{2}}\right]}{2}} \quad (160)$$

Conclusion

In this paper we have focused on the characterization properties of the Gaussian function which will be useful to a large audience. The Gaussian integral is analytically presented and used to estimate the instantaneous rate of death for an exponentially distributed mortality. An observable problem in actuarial risk is the intensity of mortality describing instantaneous rate of mortality at a specified age measured on an annual basis. The actual mortality intensity is often unknown as the true curve of death can be modeled from available mortality data. The varying degrees of mortality uncertainties and the quest for evaluating certain improper integral are therefore responsible for the emergence of approximating models to provide solution for sinusoidal problem. Furthermore, the Gaussian function has been successfully applied to present insights into sinusoidal sine and complimentary sine transform. Part of the motivation in using the Gaussian is its ability to permit an alternative technique to obtain analytically useful model to evaluate certain complex but useful improper integrals.

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