ANALYTICAL SOLUTION OF ONE-DIMENSIONAL COLLOIDAL TRANSPORT IN RIVERBANK FILTRATION SYSTEM

[1] A. D. ABUBAKAR, [2] R. O. OLAYIWOLA, [3] A. A. MOHAMMED & [4] A. T. COLE
[1] Department of Mathematics,
Federal University Gashu'a Yobe State, Nigeria
[2,3,4] Department of Mathematics,
Federal University of Technology Minna, Nigeria
E-mail: ahmeddauda8@gmail.com **Phone No:** +234-706-823-3938

Abstract

A mathematical model that described one-dimensional cotransport of colloids in riverbank filtration system, taking into account decay rate of virus is presented. The governing equations were solved analytically using parameter expanding method and eigenfunction expansion techniques, while the results obtained were presented graphically and discussed. The results obtained revealed that the virus concentration increases with increase in the value of decay rate and later decreases. Also, the hydrodynamic dispersion coefficient decreases both the concentration of virus and dissolved organic matter (DOM).

Keywords: Analytical solution, colloids, riverbank filtration.

Introduction

It has been reported by Kallioras *et al.* (2006) that aquifers occupy 2.5% of freshwater on earth and only less than 1% of Earth's water can be found in rivers, lakes, or atmosphere layers. Many countries depend heavily on river water as a source of agricultural and drinking water purposes despite this small percentage (Ad de *et al.*, 2012). Riverbank filtration (RBF) is a natural technology that is use to remove pathogens, natural organic matter, dissolve organic matter (DOM) etc in the river water as the water moves to the aquifer through the riverbed. Singh (2017) reported that the biggest challenge in 21st century is water security for every individual. The demand and the degree of contamination in raw water sources all around the world have significantly increased as a result of increase in population growth, agricultural and industrial activities.

However, the presence of contaminants in rivers may cause detrimental effects on the environment, human health and crop productivity (Schwarzenbach *et al.*, 2010). Consequently, different diseases that can be fatal for individuals may occur. For example, the usage of agriculture fertilizers can cause contamination of river water by nitrates chemicals (Kowal & Polik, 1987). These compounds have harmful effects on human health, especially for infants, young children, elderly individuals, pregnant and nursing women (U.S. Environmental Protection Agency, 2009). For infants, the nitrate compounds can cause blue baby syndrome where the blood cannot properly carry oxygen (U.S. Environmental Protection Agency, 2009). This situation leads to infant death if there is no immediate medical attention (Schwarzenbach *et al.*, 2010). The main symptom of the presence of pathogens in poor quality water is diarrhea. It causes 1.8 million deaths every year (WHO, 2006). Most of the enteric pathogens that causes gastrointestinal illnesses, do not live indefinitely in water. Hence, the concentration of these pathogens in wastewater determines the types of disease in the community (WHO, 2017). The overall design of RBF systems requires detailed hydrogeological site investigation, knowledge about the hydrological characteristics of the catchment as well as defining the catchment area (Grischek *et al.*, 2002).

Olayiwola *et al.* (2013) presented a mathematical study of contaminant transport with first order decay and time-dependent source concentration in an aquifer. The results obtained show that contaminant concentration decreases along spatial direction as initial dispersion coefficient increases. Exact solution of contaminant transport for pumping well in riverbank filtration system, taking into account a constant pumping rate was investigated using Green's function by Mustafa *et al.* (2016).

Shende & Chau (2019) developed a model combined with hydrogeological data to determine the permissible limit of fecal coliform in water during RBF. They combined an analytical solution of water flow and contaminant transport from a stream to an infiltration gallery. From their study they observed that the analytical element method (AEM) based LIFI-PATRAM model is capable of determining the optimal location and therefore can be applied to various field problems by incorporating different properties of the aquifer and source water quality for sustainable management of RBF. It was also observed that the safe distance of an infiltration gallery is governed by the logistic function which can be effectively applied to compute the desired log cycle reduction in the concentration of pathogens during RBF system. Das *et al.* (2017) developed two analytical models for one-dimensional solute transport in the semi-infinite domain with the distance-dependent and time-dependent dispersivities. The level of the contaminant concentration was predicted for the aquifer and aquitard. The results obtained showed that the concentration distribution for different geological formations, such as aquifer and aquitard, with varying velocity field are less in aquifer than in aquitard. Jimoh *et al.* (2017) used Bubnov-Galerkin weighted residual method to study one-dimensional contaminant flow in a finite medium. Their results showed that the contaminant concentration decreases with increase in the distance from the origin as the dispersion and velocity coefficient decreases. Chatterjee and Singh (2017) presented a numerical solution for two-dimensional advection-dispersion equation with depth-dependent variable source concentration. Their results showed that the peak of contaminant concentration can be reduced significantly after a certain distance and it may be further reduced to a constant value.

Gutierrez *et al.* (2017) investigated the potential of using RBF for the highly turbid and contaminated waters in Columbia, putting into consideration the improvement of water quality and the influence of clogging through suspended solids. The results from their study showed that RBF is an appropriate technology for the removal of high turbidity, pathogens, inorganic, organic and micro-pollutant. In addition, RBF serve as protection against shock load. Selamat *et al.* (2019) gave an analysis on how bacteria can be removed and also the reduction of bacteria concentration with low frequency electromagnetic field (LF-EMF) as a component of the non-ionising radiations in RBF. They designed and constructed a LF-EMF device on horizontal coiled columns which produce uniform magnetic fields of 50 Hz. A magnetic field density was varied at 2, 4, 6, 8, and 10 mT. The results from the study indicated that bacteriain the sample of water that was exposed to the LF-EMF was statistically significantly decreased. The magnetic intensity of the LF-EMF changed the characteristic responses for bacteria. Adeboye *et al.* (2013) studied the optimal analysis of contaminant inversion in an unconfined aquifer system. The result of their study shows that the level of contaminant reduces as time increases in the domain.

The present work seeks to investigate the effect of decay rate on virus concentration in RBF while considering DOM as a nutrient for bacteria.

Mathematical Formulation of the Problem

We consider unconfined, homogeneous and isotropic aquifer. The origin is considered as the point source of colloidal concentration while at initial time, the colloidal concentration is not

uniform. It is assumed that the aquifer is saturated with dissolve organic matter (DOM) which is utilized as a nutrient for bacteria. The aquifer is assumed to have contaminants, dissolve organic matter, bacteria and virus under simplistic river bank filtration conditions. Based on the above assumptions, the equations governing the phenomenon are as follows:
The mass balance equation for bacteria (captured and suspended) in the aqueous phase of saturated porous media may be describe as:

$$\varepsilon \frac{\partial C_b}{\partial t} + K_b \rho_b \frac{\partial C_b}{\partial t} + \varepsilon V_p \frac{\partial C_b}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D_b \frac{\partial C_b}{\partial x} \right) + Q_{gmb} - Q_{dmb} + Q_{gib} - Q_{dib} + Q_{gob} \tag{2.1}$$

The mass balance equation for virus (captured and suspended) in the aqueous phase of saturated porous media may be describe as:

$$\varepsilon \frac{\partial C_v}{\partial t} + K_b \rho_b \frac{\partial C_v}{\partial t} + \varepsilon V_p \frac{\partial C_v}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D_v \frac{\partial C_v}{\partial x} \right) + Q_{gmv} - Q_{dmv} + Q_{giv} - Q_{div} + Q_{gov} \tag{2.2}$$

The mass balance equation for contaminant may be expressed as:

$$R_c \frac{\partial C_c}{\partial t} + V_p \frac{\partial C_c}{\partial x} = \frac{\partial}{\partial x}\left(D_c \frac{\partial C_c}{\partial x} \right) - \frac{\mu_{\max} C_b}{K_s Y_b}(1 + \rho_s K_1 + \rho_b K_b) C_c - \frac{\mu_{\max} C_v}{K_s Y_v}(1 + \rho_s K_1 + \rho_v K_v) C_c + m_o(x,\, t) \tag{2.3}$$

The mass balance equation for DOM in the aqueous phase may be expressed as:

$$\varepsilon \frac{\partial C_o}{\partial t} + \rho_s K_0 \frac{\partial C_o}{\partial t} + \varepsilon V_p \frac{\partial C_o}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D_o \frac{\partial C_o}{\partial x} \right) - Q_{oc} \tag{2.4}$$

If bacteria and virus can utilize contaminants sorbed on bacteria, virus, DOM and solid matrix, the growth rates of mobile bacteria and mobile virus with contaminants as a food source may be presented respectively as (Kim & Corupcougla, 1996):

$$Q_{gmb} = \frac{\mu_{\max}}{K_s}\left(C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + C_o K_2 C_c + C_b K_3 C_c + C_v K_5 C_c \right) \varepsilon C_b \tag{2.5}$$

$$Q_{gmv} = \frac{\mu_{\max}}{K_s}\left(C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + C_o K_2 C_c + C_b K_3 C_c + C_v K_5 C_c \right) \varepsilon C_v \tag{2.6}$$

The growth rates of immobile bacteria and immobile virus may be presented respectively as:

$$Q_{gib} = \frac{\mu_{\max}}{K_s}\left(C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + \frac{\rho_b K_b C_b K_4 C_c}{\varepsilon} + \frac{\rho_v K_v C_v K_6 C_c}{\varepsilon} \right) \rho_b K_b C_b \tag{2.7}$$

$$Q_{giv} = \frac{\mu_{\max}}{K_s}\left(C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + \frac{\rho_b K_b C_b K_4 C_c}{\varepsilon} + \frac{\rho_v K_v C_v K_6 C_c}{\varepsilon} \right) \rho_v K_v C_v \tag{2.8}$$

The growth rates of bacteria with DOM as a food source may be expressed as first-order kinetic expression as (Borden & Bedient, 1986):

$$Q_{gob} = K_o Y_b \varepsilon C_o \tag{2.9}$$

The decay rates of mobile bacteria and mobile virus may be expressed as first-order kinetic expression respectively as (Borden & Bedient, 1986):

$$Q_{dmb} = K_{dmb} \varepsilon C_b \tag{2.10}$$

$$Q_{dmv} = K_{dmv} \varepsilon C_v \tag{2.11}$$

The decay rates of immobile bacteria and immobile virus may be expressed as first-order kinetic expression respectively as:

$$Q_{dib} = K_{dib} \rho_b K_b C_b \tag{2.12}$$

$$Q_{div} = K_{div}\rho_v K_v C_v \qquad (2.13)$$

Equation (2.14) represents the utilization rates of contaminants sorbed on DOM.

$$Q_{oc} = \frac{\mu_{\max} C_o K_2 C_c}{K_s Y_b}(\varepsilon C_b) \qquad (2.14)$$

Where:

β is the biological processes at the riverbed, R_c is the retardation factor, q is the stream depletion flow rate, S_s is the specific storage coefficient, K is the hydraulic conductivities, $m_o(x, t)$ is the concentration of contaminant measured at the pumping well, Q_o is the pumping rate, L is the distance of the pumping well from the river, C_b, C_v, C_c & C_o are the concentration of the bacteria, virus, contaminant and DOM suspended in the aqueous phase respectively, ε is the water content, V_p is the pore water velocity, D_b, D_v, D_c & D_o are the hydrodynamic dispersion coefficient of bacteria, virus, contaminant and DOM respectively, ρ_b & ρ_v are the density of bacteria and virus respectively, σ_b & σ_v are the volumetric fraction of bacteria and virus respectively, K_b, K_v are the linear equilibrium distribution coefficient of bacteria and virus between the aqueous phase and the solid phase respectively, K_0 is the first-order decay rate coefficient of DOM, K_s is the half saturation constant, Y_b *and* Y_v is the yield coefficient of bacteria and virus, $\mu_{\max}$ is the maximum growth rate, K_1 is the linear equilibrium distribution coefficient of contaminants between the aqueous phase and the solid matrix, K_2 is the linear equilibrium distribution coefficient of contaminants between the aqueous phase and DOM, K_3 & K_4 are the linear equilibrium distribution coefficient of contaminants between the aqueous phase, mobile and immobile bacteria respectively, K_5 & K_6 are the linear equilibrium distribution coefficient of contaminants between the aqueous phase, mobile and immobile virus respectively, K_{dmb} & K_{dib} are the decay rate coefficient of mobile and immobile bacteria respectively, K_{dmv} & K_{div} are the decay rate coefficient of mobile and immobile virus respectively, ρ_s is the dry bulk density of solid matrix,

σ_{cs} is the mass fraction of contaminants sorbed on solid matrix, Q_{oc} is the utilization rate of contaminants sorbed on DOM, σ_{co} is the mass fraction of contaminants attached to DOM.

Substituting equation (2.5) – (2.14) in to (2.1) – (2.4), we obtain

$$\varepsilon\frac{\partial C_b}{\partial t} + \rho_b K_b \frac{\partial C_b}{\partial t} + \varepsilon V_p \frac{\partial C_b}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D_b \frac{\partial C_b}{\partial x}\right) + \frac{\mu_{\max}}{K_s}\left(\begin{array}{c} C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + C_o K_2 C_c + \\ C_b K_3 C_c + C_v K_5 C_c \end{array}\right)\varepsilon C_b -$$

$$K_{dmb}\varepsilon C_b + K_o Y_b \varepsilon C_o + \frac{\mu_{\max}}{K_s}\left(\begin{array}{c} C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + \frac{\rho_b K_b C_b K_4 C_c}{\varepsilon} + \\ \frac{\rho_v K_v C_v K_6 C_c}{\varepsilon} \end{array}\right)\rho_b K_b C_b - K_{dib}\rho_b K_b C_b \qquad 2.15)$$

$$\varepsilon \frac{\partial C_v}{\partial t} + \rho_v K_v \frac{\partial C_v}{\partial t} + \varepsilon V_p \frac{\partial C_v}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D_v \frac{\partial C_v}{\partial x} \right) +$$
$$\frac{\mu_{\max}}{K_s}\left(C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + C_o K_2 C_c + C_b K_3 C_c + C_v K_5 C_c \right)\varepsilon C_v - K_{dmv}\varepsilon C_v + \tag{2.16}$$
$$\frac{\mu_{\max}}{K_s}\left(C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + \frac{\rho_b K_b C_b K_4 C_c}{\varepsilon} + \frac{\rho_v K_v C_v K_6 C_c}{\varepsilon} \right)\rho_v K_v C_v - K_{div}\rho_v K_v C_v$$

$$R_c \frac{\partial C_c}{\partial t} + V_p \frac{\partial C_c}{\partial x} = \frac{\partial}{\partial x}\left(D_c \frac{\partial C_c}{\partial x} \right) - \frac{\mu_{\max} C_b}{K_s Y_b}(1 + \rho_s K_1 + \rho_b K_b) C_c -$$
$$\frac{\mu_{\max} C_v}{K_s Y_v}(1 + \rho_s K_1 + \rho_v K_v) C_c + \frac{q}{Q_o\left(1 - \frac{x}{L + x} \right)} S_s e^{-\frac{\beta}{R_c} t} \tag{2.17}$$

$$\varepsilon \frac{\partial C_o}{\partial t} + \rho_s K_0 \frac{\partial C_o}{\partial t} + \varepsilon V_p \frac{\partial C_o}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D_o \frac{\partial C_o}{\partial x} \right) - \frac{\mu_{\max} C_o K_2 C_{co}}{K_s Y_b} \varepsilon C_b \tag{2.18}$$

The dependence of diffusion coefficients on the concentration of contaminant is taken in to account by the mathematical expressions:

$$D_b = D_{bo} e^{\sigma C_c}, \qquad D_v = D_{vo} e^{\eta C_c}, \qquad D_c = D_{co} e^{\nu C_c} \quad \text{and} \quad D_o = D_{oo} e^{\mu C_c} \tag{2.19}$$

The initial and boundary conditions associated with the equations are formulated as:

$$\left.\begin{array}{lll}
C_b(x, 0) = \dfrac{C_{b0} x}{L}, & -D_b^* \left.\dfrac{\partial C_b}{\partial x}\right|_{x=0} + C_b(0, t) = 0, & \left.\dfrac{\partial C_b}{\partial x}\right|_{x=L} = 0 \\
C_v(x, 0) = \dfrac{C_{v0} x}{L}, & -D_v^* \left.\dfrac{\partial C_v}{\partial x}\right|_{x=0} + C_v(0, t) = 0, & \left.\dfrac{\partial C_v}{\partial x}\right|_{x=L} = 0 \\
C_c(x, 0) = \dfrac{C_{c0} x}{L}, & -D_c^* \left.\dfrac{\partial C_c}{\partial x}\right|_{x=0} + C_c(0, t) = 0, & \left.\dfrac{\partial C_c}{\partial x}\right|_{x=L} = 0 \\
C_o(x, 0) = \dfrac{C_{o0} x}{L}, & -D_o^* \left.\dfrac{\partial C_o}{\partial x}\right|_{x=0} + C_o(0, t) = 0, & \left.\dfrac{\partial C_o}{\partial x}\right|_{x=L} = 0
\end{array}\right\} \tag{2.20}$$

Method of Solution

Dimensional Analysis

Equation (2.15) - (2.20) were non-dimensionalized using the following dimensionless variables

$$t' = \frac{Kt}{S_s L^2}, \; x' = \frac{x}{L}, \; \phi = \frac{C_b}{C_{bo}}, \; \psi = \frac{C_v}{C_{vo}}, \quad \theta = \frac{C_c}{C_{co}}, \quad \varphi = \frac{C_o}{C_{oo}} \tag{2.21}$$

Then we obtain

$$R_b \frac{\partial \phi}{\partial t} + \alpha_0 \frac{\partial \phi}{\partial x} = D_1 \frac{\partial}{\partial x}\left((1 + \sigma_1 \theta) \frac{\partial \phi}{\partial x} \right) - \alpha_1 \phi + \alpha_4 (\alpha_5 \theta + \alpha_6 \varphi \theta + \alpha_7 \phi \theta + \alpha_8 \psi \theta) \phi +$$
$$\alpha_9 (\alpha_5 \theta + \beta_1 \phi \theta + \beta_2 \psi \theta) \phi + \alpha_3 \varphi \tag{2.22}$$

$$R_v \frac{\partial \psi}{\partial t} + \alpha_0 \frac{\partial \psi}{\partial x} = D_2 \frac{\partial}{\partial x}\left((1+\eta_1\theta)\frac{\partial \psi}{\partial x}\right) - r_0\psi + r_1\begin{pmatrix} \alpha_5\theta + \alpha_6\varphi\theta + \\ \alpha_7\phi\theta + \alpha_8\psi\theta \end{pmatrix}\psi \tag{2.23}$$
$$+ r_2(\alpha_5\theta + \beta_1\phi\theta + \beta_2\psi\theta)\psi$$

$$R_c \frac{\partial \theta}{\partial t} + \alpha_0 \frac{\partial \theta}{\partial x} = D_3 \frac{\partial}{\partial x}\left((1+\nu_1\theta)\frac{\partial \theta}{\partial x}\right) - m_1\phi\theta - m_2\psi\theta + \frac{m_3}{\left(1-\dfrac{x}{1+x}\right)}e^{-\lambda_3 t} \tag{2.24}$$

$$R_o \frac{\partial \varphi}{\partial t} + \alpha_0 \frac{\partial \varphi}{\partial x} = D_4 \frac{\partial}{\partial x}\left((1+\mu_1\theta)\frac{\partial \varphi}{\partial x}\right) - g_1\phi\varphi\theta \tag{2.25}$$

$$\left.\begin{array}{lll}
\phi(x,0) = x, & -D_1^* \left.\dfrac{\partial \phi}{\partial x}\right|_{x=0} + \phi(0,t) = 0, & \left.\dfrac{\partial \phi}{\partial x}\right|_{x=1} = 0 \\
\psi(x,0) = x, & -D_2^* \left.\dfrac{\partial \psi}{\partial x}\right|_{x=0} + \psi(0,t) = 0, & \left.\dfrac{\partial \psi}{\partial x}\right|_{x=1} = 0 \\
\theta(x,0) = x, & -D_3^* \left.\dfrac{\partial \theta}{\partial x}\right|_{x=0} + \theta(0,t) = 0, & \left.\dfrac{\partial \theta}{\partial x}\right|_{x=1} = 0 \\
\varphi(x,0) = x, & -D_4^* \left.\dfrac{\partial \varphi}{\partial x}\right|_{x=0} + \varphi(0,t) = 0, & \left.\dfrac{\partial \varphi}{\partial x}\right|_{x=1} = 0
\end{array}\right\} \tag{2.26}$$

Analytical Solution

Equations (2.22) – (2.25) satisfies (2.26) were solved analytically using parameter expanding method and eigenfunction expansion technique and we obtained

$$\phi(x,t) = e^{p_{10}t} + \sum_{n=1}^{\infty} p_8 e^{p_{11}t}\cos n\pi x + \frac{f_o}{D_1^*}\left(e^{p_{10}t} + \sum_{n=1}^{\infty} p_8 e^{p_{11}t}\right)\left(x - \frac{x^2}{2}\right) + \\ L_o(t) + \sum_{n=1}^{\infty} L_n(t)\cos n\pi x \tag{2.27}$$

$$\psi(x,t) = e^{p_{10}t} + \sum_{n=1}^{\infty} p_8 e^{p_{11}t}\cos n\pi x + \frac{f_o}{D_2^*}\left(e^{p_7 t} + \sum_{n=1}^{\infty} p_8 e^{p_9 t}\right)\left(x - \frac{x^2}{2}\right) + \\ Q_o(t) + \sum_{n=1}^{\infty} Q_n(t)\cos n\pi x \tag{2.28}$$

$$\theta(x,t) = p_2 - p_3 e^{-\lambda_3 t} + \sum_{n=1}^{\infty}\left(p_4 e^{-\lambda_3 t} + p_5 e^{-p_6 t}\right)\cos n\pi x + \\ \frac{f_o}{D_3^*}\left(p_2 - p_3 e^{-\lambda_3 t} + \sum_{n=1}^{\infty}\left(p_4 e^{-\lambda_3 t} - p_5 e^{-p_6 t}\right)\right)\left(x - \frac{x^2}{2}\right) + N_o(t) + \sum_{n=1}^{\infty} N_n(t)\cos n\pi x \tag{2.29}$$

$$\varphi(x,t) = e^{p_{10}t} + \sum_{n=1}^{\infty} p_8 e^{p_{11}t}\cos n\pi x + \frac{f_o}{D_4^*}\left(1 + \sum_{n=1}^{\infty} p e^{-p_1 t}\right)\left(x - \frac{x^2}{2}\right) + \\ M_o(t) + \sum_{n=1}^{\infty} M_n(t)\cos n\pi x \tag{2.30}$$

Where,

$$
L_o(t)=e^{p_{10}t}\left(\frac{2f_0\alpha_1}{3R_bD_1^*}-\frac{2f_0D_1}{R_bD_1^*}\right)\left(t+\sum_{n=1}^{\infty}\frac{p_8\left(e^{(p_{11}-p_{10})t}-1\right)}{p_{11}-p_{10}}\right)-\frac{2f_0e^{p_{10}t}}{3D_1^*}\left(p_{10}t+\sum_{n=1}^{\infty}\frac{p_8p_{11}\left(e^{(p_{11}-p_{10})t}-1\right)}{p_{11}-p_{10}}\right)+
$$

$$
\sum_{n=1}^{\infty}\frac{\left((-1)^n-1\right)p_8\left(e^{(p_{11}-2p_{10})t}-e^{p_{10}t}\right)}{R_b(p_{11}-p_{10})}
$$

$$
+\frac{2e_1e^{p_{10}t}}{R_b}\left(p_2t+\frac{p_3\left(e^{-\lambda_3t}-1\right)}{\lambda_3}+\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{(p_{11}-p_{10}-\lambda_3)t}-1\right)}{2(p_{11}-p_{10}-\lambda_3)}+\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{(p_{11}-p_{10}-p_6)t}-1\right)}{2(p_{11}-p_{10}-p_6)}\right)
$$

$$
+\frac{2e_2e^{p_{10}t}}{R_v}\left(\begin{aligned}&\frac{p_2\left(e^{p_{10}t}-1\right)}{p_{10}}-\frac{p_3\left(e^{(p_{10}-\lambda_3)t}-1\right)}{p_{10}-\lambda_3}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_2p_8^2(e^{(2p_{11}-p_{10})t}-1)}{2(2p_{11}-p_{10})}-\\&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_3p_8^2\left(e^{(2p_{11}-p_{10}-\lambda_3)t}-1\right)}{2(2p_{11}-p_{10}-\lambda_3)}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{(p_{11}-\lambda_3)t}-1\right)}{2(p_{11}-\lambda_3)}\\&+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{(p_{11}+p_7-p_{10}-p_6)t}-1\right)}{2(p_{11}+p_7-p_{10}-p_6)}\end{aligned}\right)-
$$

$$
+\frac{2e_3e^{p_{10}t}}{R_b}\left(\begin{aligned}&\frac{p_2\left(e^{p_7t}-1\right)}{p_7}-\frac{p_3\left(e^{(p_7-\lambda_3)t}-1\right)}{p_7-\lambda_3}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{(p_{11}+p_7-p_{10}-\lambda_3)t}-1\right)}{2(p_{11}+p_7-p_{10}-\lambda_3)}+\\&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{(p_{11}+p_7-p_{10}-p_6)t}-1\right)}{2(p_{11}+p_7-p_{10}-p_6)}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{(p_9-\lambda_3)t}-1\right)}{2(p_9-\lambda_3)}\\&+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{(p_9-p_6)t}-1\right)}{2(p_9-p_6)}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_2p_8^2\left(e^{(p_{11}+p_9-p_{10}-\lambda_3)t}-1\right)}{2(p_{11}+p_9-p_{10}-\lambda_3)}-\\&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_3p_8^2\left(e^{(p_{11}+p_9-p_{10}-p_6)t}-1\right)}{2(p_{11}+p_9-p_{10}-p_6)}\end{aligned}\right)
$$

$$
+\frac{2b_5\alpha_6e^{p_{10}t}}{R_b}\left(\begin{aligned}&p_2t+\frac{p_3\left(e^{-\lambda_3t}-1\right)}{\lambda_3}-\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{-(p_1+\lambda_3)t}-1\right)}{2(p_1+\lambda_3)}-\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{-(p_1+p_6)t}-1\right)}{2(p_1+p_6)}\\&+\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{(p_{11}-p_{10}-\lambda_3)t}-1\right)}{2(p_{11}-p_{10}-\lambda_3)}+\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{(p_{11}-p_{10}-p_6)t}-1\right)}{2(p_{11}-p_{10}-p_6)}\\&+\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_2p_8^2\left(e^{(p_{11}-p_{10}-p_1)t}-1\right)}{2(p_{11}-p_{10}-p_1)}-\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_3p_8^2\left(e^{(p_{11}-p_{10}-p_1-\lambda_3)t}-1\right)}{2(p_{11}-p_{10}-p_1-\lambda_3)}\end{aligned}\right)
$$

$$
-\frac{2b_7\left(1-e^{p_{10}t}\right)}{R_bp_{10}}+\frac{2n_4e^{p_{10}t}}{3}
$$

$$L_n(t) = -\frac{2f_0\alpha_1 e^{p_{11}t}}{R_b D_1^*(n\pi)^2}\left(\frac{e^{(p_{10}-p_{11})t}}{p_{10}-p_{11}} + \sum_{n=1}^{\infty} p_8 t\right) + \frac{2f_0 e^{p_{10}t}}{D_1^*(n\pi)^2}\left(\sum_{n=1}^{\infty} p_8 p_{11} t + \frac{p_{10}\left(e^{(p_{10}-p_{11})t}-1\right)}{p_{10}-p_{11}}\right)$$

$$-\frac{D_1 b_4 e^{p_{11}t}}{R_b}\left(\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_3 p_8 (n\pi)^2\left(e^{-\lambda_3 t}-1\right)}{\lambda_3} + \sum_{n=1}^{\infty}(n\pi)^2 p_2 p_8 t\right) + \sum_{n=1}^{\infty}\frac{\left((-1)^{2n}-1\right)p_8 t e^{p_{11}t}}{2R_b}$$

$$+\frac{e_1 e^{p_{11}t}}{R_b}\left(\sum_{n=1}^{\infty} p_2 p_8 t + \sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{-\lambda_3 t}-1\right)}{\lambda_3} + \sum_{n=1}^{\infty}\frac{p_4\left(e^{(p_{10}-p_{11}-\lambda_3)t}-1\right)}{(p_{10}-p_{11}-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_5\left(e^{(p_{10}-p_{11}-p_6)t}-1\right)}{(p_{10}-p_{11}-p_6)}\right)$$

$$+\frac{2e_2 e^{p_{11}t}}{R_b}\left(\begin{aligned}&\sum_{n=1}^{\infty}\frac{p_2 p_8\left(e^{p_7 t}-1\right)}{p_7} - \sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{(p_{10}-\lambda_3)t}-1\right)}{p_{10}-\lambda_3} + \sum_{n=1}^{\infty}\frac{p_4\left(e^{(2p_{10}-p_{11}-\lambda_3)t}-1\right)}{2(2p_{10}-p_{11}-\lambda_3)} + \\ &\sum_{n=1}^{\infty}\frac{p_4\left(e^{(2p_{10}-p_{11}-\lambda_3)t}-1\right)}{2(2p_{10}-p_{11}-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_5\left(e^{(2p_{10}-p_{11}-p_6)t}-1\right)}{2(2p_{10}-p_{11}-p_6)} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{3p_4 p_8^2\left(e^{(p_{11}-\lambda_3)t}-1\right)}{8(p_{11}-\lambda_3)} \\ &+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{3p_5 p_8^2\left(e^{(p_{11}-p_6)t}-1\right)}{8(p_{11}-p_6)}\end{aligned}\right)$$

$$+\frac{2e_3 e^{p_{11}t}}{R_b}\left(\begin{aligned}&\sum_{n=1}^{\infty}\frac{p_2 p_8\left(e^{p_7 t}-1\right)}{2p_7} - \sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{(p_7-\lambda_3)t}-1\right)}{2(p_7-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_4\left(e^{(p_{10}+p_7-p_{11}-\lambda_3)t}-1\right)}{2(p_{10}+p_7-p_{11}-\lambda_3)} \\ &+\sum_{n=1}^{\infty}\frac{p_5\left(e^{(p_{10}+p_7-p_{11}-p_6)t}-1\right)}{2(p_{10}+p_7-p_{11}-p_6)} + \sum_{n=1}^{\infty}\frac{p_2 p_8\left(e^{(p_{10}+p_9-p_6)t}-1\right)}{2(p_{10}+p_9-p_6)} - \\ &\sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{(p_{10}+p_9-p_6-\lambda_3)t}-1\right)}{2(p_{10}+p_9-p_6-\lambda_3)} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{3p_4 p_8^2\left(e^{(p_9-\lambda_3)t}-1\right)}{8(p_9-\lambda_3)} \\ &+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{3p_5 p_8^2\left(e^{(p_9-p_6)t}-1\right)}{8(p_9-p_6)}\end{aligned}\right) +$$

$$\frac{2b_5\alpha_6 e^{p_{11}t}}{R_b}\left(\begin{aligned}&\sum_{n=1}^{\infty}\frac{p_4\left(e^{(p_{10}-p_{11}-\lambda_3)t}-1\right)}{2(p_{10}-p_{11}-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_5\left(e^{(p_{10}-p_{11}-p_6)t}-1\right)}{2(p_{10}-p_{11}-p_6)} - \sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{(p_1-p_{11}-\lambda_3)t}-1\right)}{2(p_1-p_{11}-\lambda_3)} \\ &+\sum_{n=1}^{\infty}\frac{p_2 p_8\left(e^{(p_{10}-p_{11}-p_1)t}-1\right)}{2(p_{10}-p_{11}-p_1)} - \sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{(p_{10}-p_{11}-p_1-\lambda_3)t}-1\right)}{2(p_{10}-p_{11}-p_1-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_2 p_8 t}{2} \\ &-\sum_{n=1}^{\infty}\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{3p_4 p_8^2\left(e^{-(p_1+\lambda_3)t}-1\right)}{8(p_1+\lambda_3)} - \sum_{n=1}^{\infty}\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{3p_5 p_8^2\left(e^{-(p_1+p_6)t}-1\right)}{8(p_1+p_6)}\end{aligned}\right)$$

$$-\sum_{n=1}^{\infty}\frac{b_7 p_8\left(e^{-(p_1+2p_{11})t}-e^{p_{11}t}\right)}{R_b(p_{11}+p_1)} - \frac{2n_4 e^{p_{11}t}}{(n\pi)^2}$$

$$Q_o(t)=e^{p_7 t}\left(\frac{2f_0r_0}{3R_vD_2^*}-\frac{2f_0D_2}{R_vD_2^*}\right)\left(t+\sum_{n=1}^{\infty}\frac{p_8\left(e^{(p_9-p_7)t}-1\right)}{p_9-p_7}\right)-\frac{2f_0e^{p_7 t}}{3D_2^*}\left(p_7t+\sum_{n=1}^{\infty}\frac{p_8p_9\left(e^{(p_9-p_7)t}-1\right)}{p_9-p_7}\right)$$

$$+\sum_{n=1}^{\infty}\frac{\left((-1)^n-1\right)p_8\left(e^{(p_9-2p_7)t}-e^{p_9 t}\right)}{R_v(p_9-p_7)}$$

$$+\frac{2e_4e^{p_7 t}}{R_v}\left(p_2t+\frac{p_3\left(e^{-\lambda_3 t}-1\right)}{\lambda_3}+\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{pp_4\left(e^{(p_9-p_7-\lambda_3)t}-1\right)}{2(p_9-p_7-\lambda_3)}+\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{pp_5\left(e^{(p_9-p_7-p_6)t}-1\right)}{2(p_9-p_7-p_6)}\right)$$

$$+\frac{2e_5e^{p_7 t}}{R_v}\left(\begin{array}{l}\frac{p_2\left(e^{p_{10}t}-1\right)}{p_{10}}-\frac{p_3\left(e^{(p_{10}-\lambda_3)t}-1\right)}{p_{10}-\lambda_3}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8(e^{(p_{10}+p_9-p_7-\lambda_3)t}-1)}{2(p_{10}+p_9-p_7-\lambda_3)}+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{(p_{10}+p_9-p_7-p_6)t}-1\right)}{2(p_{10}+p_9-p_7-p_6)}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{(p_{11}-\lambda_3)t}-1\right)}{2(p_{11}-\lambda_3)}\\ +\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{(p_{11}-p_6)t}-1\right)}{2(p_{11}-p_6)}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_2p_8^2\left(e^{(p_{11}+p_9-p_7)t}-1\right)}{2(p_{11}+p_9-p_7)}+\\ -\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_3p_8^2\left(e^{(p_{11}+p_9-p_7-\lambda_3)t}-1\right)}{2(p_{11}+p_9-p_7-\lambda_3)}\end{array}\right)-$$

$$+\frac{2e_6e^{p_7 t}}{R_v}\left(\begin{array}{l}\frac{p_2\left(e^{p_7 t}-1\right)}{p_7}-\frac{p_3\left(e^{(p_7-\lambda_3)t}-1\right)}{p_7-\lambda_3}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_2p_8^2\left(e^{(2p_9-p_7)t}-1\right)}{2(2p_9-p_7)}-\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_3p_8^2\left(e^{(2p_9-p_7-\lambda_3)t}-1\right)}{2(2p_9-p_7-\lambda_3)}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{(p_9-\lambda_3)t}-1\right)}{(p_9-\lambda_3)}\\ +\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{(p_9-p_6)t}-1\right)}{(p_9-p_6)}\end{array}\right)$$

$$+\frac{2b_9\alpha_6e^{p_7 t}}{R_v}\left(\begin{array}{l}p_2t+\frac{p_3\left(e^{-\lambda_3 t}-1\right)}{\lambda_3}+\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{(p_9-p_7-\lambda_3)t}-1\right)}{2(p_9-p_7-\lambda_3)}+\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{(p_9-p_7-p_6)t}-1\right)}{2(p_9-p_7-p_6)}\\ -\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_4p_8\left(e^{-(p_1+\lambda_3)t}-1\right)}{2(p_1+\lambda_3)}-\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_5p_8\left(e^{-(p_1+p_6)t}-1\right)}{2(p_1+p_6)}\\ +\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_2p_8^2\left(e^{(p_9-p_7-p_1)t}-1\right)}{2(p_9-p_7-p_1)}-\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_3p_8^2\left(e^{(p_9-p_7-p_1-\lambda_3)t}-1\right)}{2(p_9-p_7-p_1-\lambda_3)}\end{array}\right)$$

$$+\frac{2n_3e^{p_7 t}}{3}$$

$$Q_n(t) = -\frac{2f_0 r_0 e^{p_9 t}}{R_v D_2^*(n\pi)^2}\left(\frac{e^{(p_9-p_7)t}}{p_9-p_7} + \sum_{n=1}^{\infty} p_8 t\right) + \frac{2f_0 e^{p_9 t}}{D_2^*(n\pi)^2}\left(\sum_{n=1}^{\infty} p_8 p_9 t + \frac{p_7\left(e^{(p_7-p_9)t}-1\right)}{p_7-p_9}\right)$$

$$+\frac{D_2 b_8 e^{p_9 t}}{R_v}\left(-\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{p_3 p_8 (n\pi)^2\left(e^{-\lambda_3 t}-1\right)}{\lambda_3} - \sum_{n=1}^{\infty}(n\pi)^2 p_2 p_8 t\right) + \sum_{n=1}^{\infty}\frac{\left((-1)^{2n}-1\right)p_8 t e^{p_9 t}}{2R_v}$$

$$+\frac{e_4 e^{p_9 t}}{R_v}\left(\sum_{n=1}^{\infty} p_2 p t + \sum_{n=1}^{\infty}\frac{p p_3\left(e^{-\lambda_3 t}-1\right)}{\lambda_3} + \sum_{n=1}^{\infty}\frac{p_4\left(e^{(p_7-p_9-\lambda_3)t}-1\right)}{(p_7-p_9-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_5\left(e^{(p_7-p_9-p_6)t}-1\right)}{(p_7-p_9-p_6)}\right)$$

$$+\frac{2e_5 e^{p_9 t}}{R_v}\left(\begin{array}{l}
\frac{p_2\left(e^{p_{10} t}-1\right)}{p_{10}} - \frac{p_3\left(e^{(p_{10}-\lambda_3)t}-1\right)}{p_{10}-\lambda_3} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_4 p_8 (e^{(p_{10}+p_9-p_7-\lambda_3)t}-1)}{2(p_{10}+p_9-p_7-\lambda_3)} + \\
\sum_{n=1}^{\infty}\frac{p_4\left(e^{(p_{10}-p_9+p_7-\lambda_3)t}-1\right)}{2(p_{10}-p_9+p_7-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_5\left(e^{(p_{10}-p_9+p_7-p_6)t}-1\right)}{2(p_{10}-p_9+p_7-p_6)} \\
+\sum_{n=1}^{\infty}\frac{p_2 p_8\left(e^{p_{10} t}-1\right)}{2p_{10}} - \sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{(p_{10}-\lambda_3)t}-1\right)}{2(p_{10}-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_2 p_8\left(e^{(p_{11}+p_7-p_9)t}-1\right)}{2(p_{11}+p_7-p_9)} - \\
-\sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{(p_{11}+p_7-p_9-\lambda_3)t}-1\right)}{2(p_{11}+p_7-p_9-\lambda_3)} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_4 p_8^2\left(e^{(p_{11}-\lambda_3)t}-1\right)}{2(p_{11}-\lambda_3)} + \\
\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{3p_5 p_8^2\left(e^{(p_{11}-p_6)t}-1\right)}{8(p_{11}-p_6)}
\end{array}\right)$$

$$+\frac{2e_6 e^{p_9 t}}{R_v}\left(\begin{array}{l}
\sum_{n=1}^{\infty}\frac{p_2 p_8\left(e^{p_7 t}-1\right)}{p_7} - \sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{(p_7-\lambda_3)t}-1\right)}{p_7-\lambda_3} + \sum_{n=1}^{\infty}\frac{p_4\left(e^{(2p_7-p_9-\lambda_3)t}-1\right)}{2(2p_7-p_9-\lambda_3)} \\
+\sum_{n=1}^{\infty}\frac{p_5\left(e^{(2p_7-p_9-p_6)t}-1\right)}{2(2p_7-p_9-p_6)} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{3p_4 p_8^2\left(e^{(p_9-\lambda_3)t}-1\right)}{8(p_9-\lambda_3)} \\
+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{3p_5 p_8^2\left(e^{(p_9-p_6)t}-1\right)}{8(p_9-p_6)}
\end{array}\right)$$

$$+\frac{2b_9 \alpha_6 e^{p_9 t}}{R_v}\left(\begin{array}{l}
\sum_{n=1}^{\infty}\frac{p_4\left(e^{(p_7-p_9-\lambda_3)t}-1\right)}{2(p_7-p_9-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_5\left(e^{(p_7-p_9-p_6)t}-1\right)}{2(p_7-p_9-p_6)} + \sum_{n=1}^{\infty}\frac{p_2 p_8 t}{2} + \sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{-\lambda_3 t}-1\right)}{2\lambda_3} \\
+\sum_{n=1}^{\infty}\frac{p_2 p_8\left(e^{(p_7-p_9-p_1)t}-1\right)}{2(p_7-p_9-p_1)} - \sum_{n=1}^{\infty}\frac{p_3 p_8\left(e^{(p_7-p_9-p_1-\lambda_3)t}-1\right)}{2(p_7-p_9-p_1-\lambda_3)} \\
-\sum_{n=1}^{\infty}\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{3p_4 p_8^2\left(e^{-(p_1+\lambda_3)t}-1\right)}{8(p_1+\lambda_3)} - \sum_{n=1}^{\infty}\sum_{n=}^{\infty}\sum_{n=1}^{\infty}\frac{3p_5 p_8^2\left(e^{-(p_1+p_6)t}-1\right)}{8(p_1+p_6)}
\end{array}\right)$$

$$-\frac{2n_3 e^{p_9 t}}{(n\pi)^2}$$

$$N_o(t) = -\frac{2f_0D_3}{R_cD_3^*}\left(p_2t + \frac{p_3(e^{-\lambda_3 t}-1)}{\lambda_3} - \sum_{n=1}^{\infty}\left(\frac{p_4(e^{-\lambda_3 t}-1)}{\lambda_3} + \frac{p_5(e^{-p_6 t}-1)}{p_6}\right)\right) -$$

$$\frac{2f_0}{3D_3^*}\left(-p_3(e^{-\lambda_3 t}-1) + \sum_{n=1}^{\infty}\left(p_4(e^{-\lambda_3 t}-1) + p_5(e^{-p_6 t}-1)\right)\right) -$$

$$\sum_{n=1}^{\infty}\frac{\left((-1)^n-1\right)}{R_c}\left(\frac{p_4(e^{-\lambda_3 t}-1)}{\lambda_3} + \frac{p_5(e^{-p_6 t}-1)}{p_6}\right) + \frac{2n_2}{3} -$$

$$\frac{2b_{12}}{R_c}\left(\begin{array}{l}\frac{p_2(e^{p_{10} t}-1)}{p_{10}} - \frac{p_3(e^{(p_{10}-\lambda_3)t}-1)}{p_{10}-\lambda_3} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{pp_4(e^{(p_{11}-\lambda_3)t}-1)}{2(p_{11}-\lambda_3)} + \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{pp_5(e^{(p_{11}-p_6)t}-1)}{2(p_{11}-p_6)}\end{array}\right) -$$

$$\frac{2b_{13}}{R_c}\left(\begin{array}{l}\frac{p_2(e^{p_7 t}-1)}{p_7} - \frac{p_3(e^{(p_7-\lambda_3)t}-1)}{p_7-\lambda_3} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{pp_4(e^{(p_9-\lambda_3)t}-1)}{2(p_9-\lambda_3)} + \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{pp_5(e^{(p_9-p_6)t}-1)}{2(p_9-p_6)}\end{array}\right)$$

$$N_n(t) = -\frac{2f_0e^{-p_6t}}{(n\pi)^2D_3^*}\left(\frac{p_3\lambda_3\left(e^{(p_6-\lambda_3)t}-1\right)}{(p_6-\lambda_3)} - \sum_{n=1}^{\infty}\left(\frac{p_3\lambda_3\left(e^{(p_6-\lambda_3)t}-1\right)}{(p_6-\lambda_3)} + p_5p_6t\right)\right) +$$

$$\frac{D_3b_{11}}{R_c}\left(\sum_{n=1}^{\infty}(n\pi)^2e^{-p_6t}\left(\frac{p_3p_4\left(e^{(p_6-2\lambda_3)t}-1\right)}{(p_6-2\lambda_3)} - \frac{p_3p_5(e^{-\lambda_3 t}-1)}{\lambda_3} - \frac{p_2p_4\left(e^{(p_6-\lambda_3)t}-1\right)}{(p_6-\lambda_3)} - p_2p_5t\right)\right) -$$

$$\sum_{n=1}^{\infty}\frac{\left((-1)^{2n}-1\right)}{2R_c}\left(\frac{p_4(e^{-\lambda_3 t}-e^{-p_6t})}{p_6-\lambda_3} + p_5te^{-p_6t}\right) - \frac{2n_2e^{-p_1t}}{(n\pi)^2} -$$

$$\frac{b_{12}e^{-p_6 t}}{R_c}\left(\begin{array}{l}\sum_{n=1}^{\infty}\frac{pp_2(e^{(p_6+p_1)t}-1)}{p_6-p_{11}} - \sum_{n=1}^{\infty}\frac{pp_3(e^{(p_6+p_{11}-\lambda_3)t}-1)}{(p_6+p_{11}-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_4(e^{(p_{10}+p_6-\lambda_3)t}-1)}{(p_{10}+p_6-\lambda_3)} + \\ \sum_{n=1}^{\infty}\frac{p_5(e^{p_{10} t}-1)}{p_{10}}\end{array}\right) -$$

$$\frac{b_{13}e^{-p_6 t}}{R_c}\left(\begin{array}{l}\sum_{n=1}^{\infty}\frac{pp_2(e^{(p_6+p_9)t}-1)}{p_6+p_2} - \sum_{n=1}^{\infty}\frac{pp_3(e^{(p_6+p_9-\lambda_3)t}-1)}{(p_6+p_9-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_4(e^{(p_6+p_7-\lambda_3)t}-1)}{(p_6+p_7-\lambda_3)} + \\ \sum_{n=1}^{\infty}\frac{p_5(e^{p_7 t}-1)}{p_7}\end{array}\right)$$

$$M_o(t) = -\frac{2f_0 D_4}{R_o D_4^*}\left(t - \sum_{n=1}^{\infty}\frac{p(e^{-p_1 t}-1)}{p_1}\right) - \frac{2f_0}{3D_o}\sum_{n=1}^{\infty}\frac{pp_1(e^{-p_1 t}-1)}{p_1} -$$

$$\sum_{n=1}^{\infty}\frac{\left((-1)^n-1\right)}{R_c}\frac{p(e^{-p_1 t}-1)}{p_1} + \frac{2n_1}{3} -$$

$$\frac{2b_{16}}{R_o}\left(\begin{array}{l} \frac{p_2(e^{p_{10}t}-1)}{p_{10}} - \frac{p_3(e^{(p_{10}+\lambda_3)t}-1)}{p_{10}-\lambda_3} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{pp_4(e^{(p_{10}-p_1-\lambda_3)t}-1)}{2(p_{10}-p_1-\lambda_3)} + \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{pp_5(e^{(p_{10}-p_1-p_6)t}-1)}{2(p_{10}-p_1-p_6)} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{pp_4(e^{(p_{11}-\lambda_3)t}-1)}{2(p_{11}-\lambda_3)} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{pp_5(e^{(p_{11}-p_6)t}-1)}{2(p_{11}-p_6)} \\ + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p^2 p_2(e^{(p_{11}-p_1)t}-1)}{2(p_{11}-p_1)} - \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{p^2 p_3(e^{(p_{11}-p_1-\lambda_3)t}-1)}{2(p_{11}-p_1-\lambda_3)} \end{array}\right)$$

$$M_n(t) = -\frac{2f_0 e^{-p_1 t}}{(n\pi)^2 D_4^*}\sum_{n=1}^{\infty} pp_1 t + \frac{2D_4 b_{15} e^{-p_1 t}}{R_o}\left(\sum_{n=1}^{\infty}\frac{pp_2(n\pi)^2 t}{2} - \sum_{n=1}^{\infty}\frac{pp_3(n\pi)^2(e^{-\lambda_3 t}-1)}{2\lambda_3}\right) -$$

$$\sum_{n=1}^{\infty}\frac{\left((-1)^{n2}-1\right)}{2R_c} pte^{-p_1 t} - \frac{2n_1 e^{-p_1 t}}{(n\pi)^2} -$$

$$\frac{2b_{16}e^{-p_1 t}}{R_o}\left(\begin{array}{l} \sum_{n=1}^{\infty}\frac{p_4(e^{(p_{10}-\lambda_3)t}-1)}{2(p_{10}-\lambda_3)} + \sum_{n=1}^{\infty}\frac{p_5(e^{(p_{10}-p_6)t}-1)}{2(p_{10}-p_6)} + \sum_{n=1}^{\infty}\frac{pp_2(e^{p_{10}t}-1)}{2p_{10}} \\ -\sum_{n=1}^{\infty}\frac{pp_3(e^{(p_{10}-\lambda_3)t}-1)}{2(p_{10}-\lambda_3)} + \sum_{n=1}^{\infty}\frac{pp_2(e^{p_{11}t}-1)}{2p_{11}} + \sum_{n=1}^{\infty}\frac{pp_3(e^{-\lambda_3 t}-1)}{2\lambda_3} \\ + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{3p^2 p_4(e^{(p_{11}-\lambda_3)t}-1)}{8(p_{11}-\lambda_3)} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{3p^2 p_5(e^{(p_{11}-p_6)t}-1)}{8(p_{11}-p_6)} \end{array}\right)$$

$$b = \frac{\alpha}{\alpha_0},\ b_1 = \frac{\alpha_2}{\alpha_0},\quad b_2 = \frac{\lambda_1}{\alpha_0},\quad b_3 = \frac{\lambda_2}{\alpha_0},\quad b_4 = \frac{\sigma_1}{\alpha_0},\quad b_5 = \frac{\alpha_4}{\alpha_0},\quad b_6 = \frac{\alpha_9}{\alpha_0},\ b_7 = \frac{\alpha_3}{\alpha_0}$$

$$b_8 = \frac{\gamma_1}{\alpha_0},\ b_9 = \frac{r_1}{\alpha_0},\ b_{10} = \frac{r_2}{\alpha_0},\ b_{11} = \frac{\nu_1}{\alpha_0},\ b_{12} = \frac{m_1}{\alpha_0},\quad b_{13} = \frac{m_2}{\alpha_0},\ b_{15} = \frac{\mu_1}{\alpha_0},\ b_{16} = \frac{g_1}{\alpha_0},$$

$$f_0 = \frac{1}{\alpha_0},\ \lambda_1 = \frac{1}{\beta_p \lambda_c},\quad m_3 = \frac{\alpha_s}{\beta_p},\quad e_1 = b_5\alpha_5 + b_6\alpha_5,\quad e_2 = b_5\alpha_7 + b_6\beta_1,\quad e_3 = b_5\alpha_8 + b_6\beta_2,$$

$$e_4 = b_9\alpha_5 + b_{10}\alpha_5,\quad e_5 = b_9\alpha_7 + b_{10}\beta_1,\quad e_6 = b_9\alpha_8 + b_{10}\beta_2,$$

$$p = p_8 = \frac{2\left((-1)^n-1\right)}{(n\pi)^2},\ p_1 = \frac{D_4(n\pi)^2}{R_o},\ p_2 = 1 + \frac{2m_3}{\lambda_3 R_c},\ p_3 = \frac{2m_3}{\lambda_3 R_c},\ p_4 = \frac{2m_3\left((-1)^n-1\right)}{R_c(n\pi)^2\left(\frac{D_3(n\pi)^2}{R_c} - \lambda_3\right)},$$

$$p_5 = \frac{2\left((-1)^n-1\right)}{(n\pi)^2}\left(1 - \frac{m_3}{R_c\left(\frac{D_3(n\pi)^2}{R_c} - \lambda_3\right)}\right),\ p_6 = \frac{D_3(n\pi)^2}{R_c},\quad p_7 = \frac{r_0}{R_v},\quad p_9 = \frac{\left(r_0 - D_2(n\pi)^2\right)}{R_v},$$

$$p_{10}=\frac{\alpha_1}{R_b},\qquad p_{11}=\frac{\left(\alpha_1-D_1(n\pi)^2\right)}{R_b},\ p_{12}=\left(\frac{(2n-1)\pi}{2}\right)^2,\ n_1=-\frac{f_0\left(1+\sum_{n=1}^{\infty}p\right)}{D_4^*}$$

$$n_2=-\frac{f_0\left(p_2-p_3+\sum_{n=1}^{\infty}(p_4+p_5)\right)}{D_3^*},\quad n_3=-\frac{f_0\left(1+\sum_{n=1}^{\infty}p_8\right)}{D_2^*},\quad n_4=-\frac{f_0\left(1+\sum_{n=1}^{\infty}p_8\right)}{D_1^*}$$

Results and Discussion

In this section, the governing equations (2.22) - (2.25) were solved analytically using parameter expanding method and eigenfunction expansions techniques, the solutions are discussed with the help of input data as follows:

From figure 1 and 2 we observed that the virus concentrations ψ increases along distance and decreases with time, but increases with increase in Decay rate r_0. Figure 3 and 4 shows that an increase in Hydrodynamic dispersion coefficient D_4, reduces DOM concentrations φ. In a similar manner, Figure 5 and 6 shows that the virus concentrations ψ increases and later decreases with an increase in dispersion coefficient values D_2.

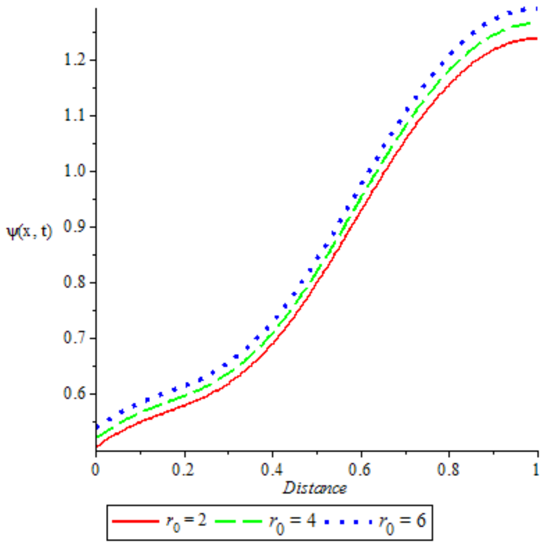

Figure 1. Relation between virus concentrations along distance at various values of decay rate coefficient

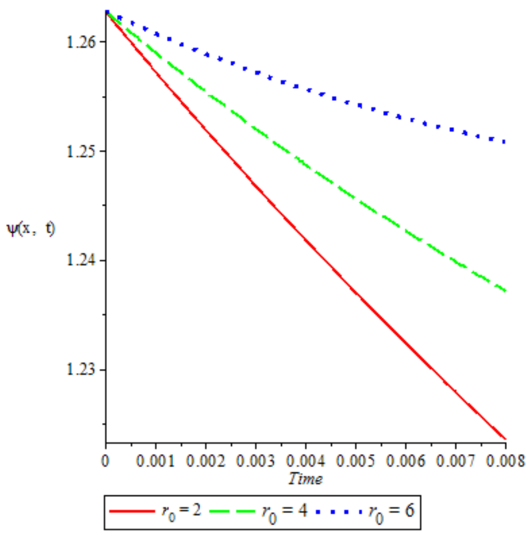

Figure 2. Relation between virus concentrations against time at various values of decay rate

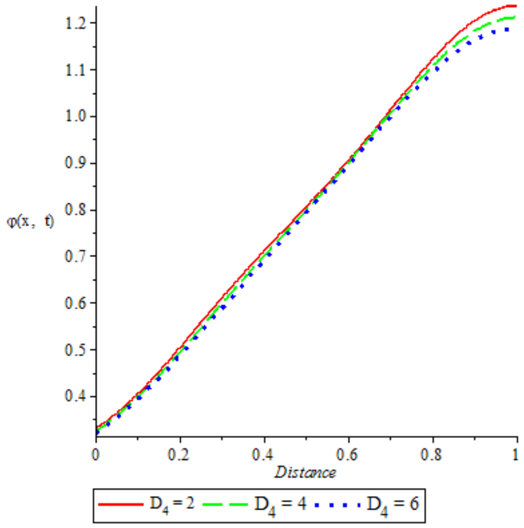

Figure 3. Relation between DOM concentrations alongdistance at various values of dispersion coefficient

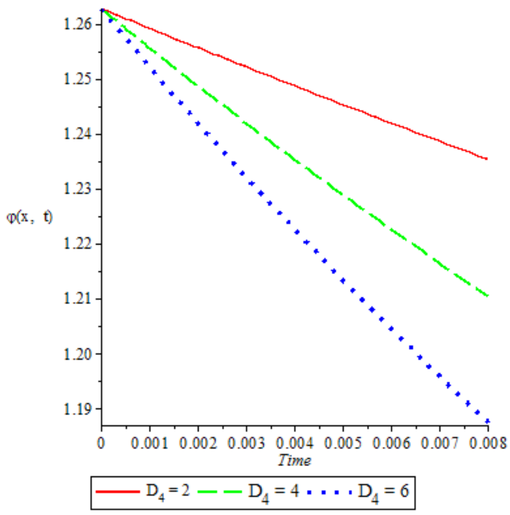

Figure 4. Relation between DOM concentrations alongtime at various values of dispersion coefficient

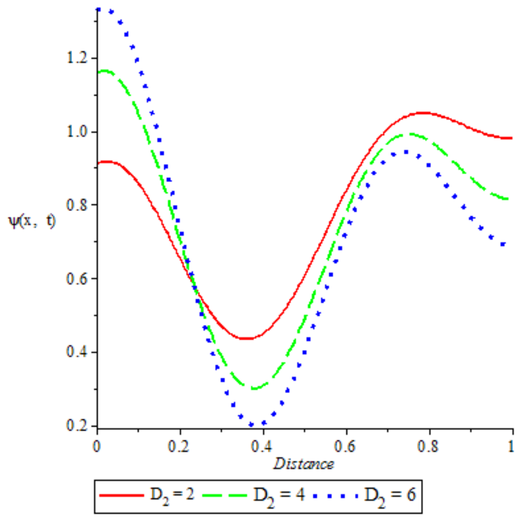

Figure 5. Relation between virus concentrations alongdistance at various values of dispersion coefficient

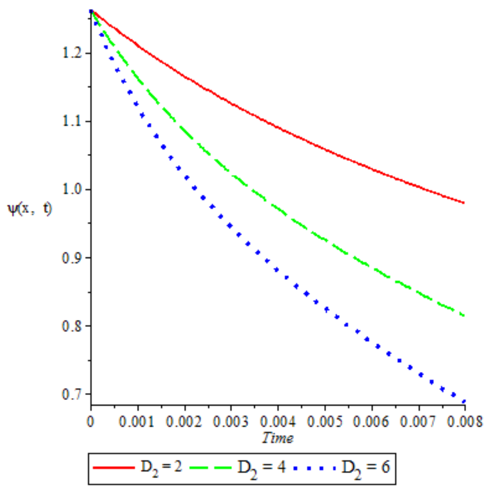

Figure 6. Relation between virus concentrations alongtime at various values of dispersion coefficient

Conclusion

In this work, we have discussed various mathematical models involving colloidal transport equations. We have also used an analytical method in evaluating the transport equations and have given profiles for different values of parameter. From the results obtained we can conclude as follows:

1. Increase in the value of decay rate, increases the concentration of virus and later reduces.
2. The hydrodynamic dispersion coefficients reduce both the concentration of virus and DOM.

3. Based on the above results, there will not be further treatment of the pumped water from RBF.

Reference

Ad de, R., Faycal, B., Peter, B., Berny, B., Ine, V., Sarah, M., Peter, S., Marco, P., Mauricio, Z., Vera, T., Alessandra, B., & Carlo, L. (2012). Current water resources in Europe and Africa - Matching water supply and water demand. Europian Scientific and Technical Research Reports. Joint research center, Institute for Environment and Sustainability, Italy. doi:10.2788/16165.

Adeboye, K. R., Shehu, M. D., & Ndanusa, A. (2013). Optimal analysis of a contaminant inversion in an unconfined aquifer system. *International Journal of Science and Technology*, 3(3), 184 – 187.

Borden, R.C., & Bedient, P. B. (1986). Transport of dissolved hydrocarbons influenced by oxygen-limited biodegradation: 1. Theoretical development. *Water Resources Research,* 22(13), 1973–1982.

Chatterjee, A., & Singh, M. K. (2017). Two-dimensional advection-dispersion equation with depth-dependent variable source concentration. *Pollution,* 4(1), 1-8.

Das, P., Begam, S., & Singh, M. K. (2017). Mathematical modeling of groundwater contamination with varying velocity field. *Journal of Hydrology and Hydromechanics,* 65(2), 192–204.

Grischek, T., Schoenheinz, D., & Ray, C. (2002). Siting and design issues for riverbank filtration schemes. Riverbank filtration: Improving source water quality. *Springer*, 291–302.

Gutirrez, J. P., Halem, D. V., & Rietveld, L. (2017). Riverbank filtration for the treatment of highly turbid Colombian rivers. *Drinking Water Engineering and Science,* 10, 13 – 26, doi:10.5194/dwes-10-13-2017.

Jimoh, O. R., Aiyesimi, Y. M., Jiya, M., & Bolarin, G. A. (2017). Semi-analytical study of one-dimensional contaminant flow in a finite medium. Journal of Applied Science Environmental Management, 21(3), 487 – 494.

Kallioras, A., Pliakas, F., & Diamantis, I. (2006). The legislative framework and policy for the water resources management of transboundary rivers in Europe: The case of Nestos/Mesta River, between Greece and Bulgaria. *Environmental Science Policy*. 9(3), 291 – 301.

Kim, S., & Corapcioglu, M.Y. (1996). A kinetic approach to modeling mobile bacteria-facilitated Ground water contaminant transport. *Water Resource Research,* 32(2), 321–331.

Kowal, A., & Polik, A. (1987). Nitrates in groundwater. In De Waal, K. and Van Den Brink, W. (Eds.) Environmental Technology. *Springer Netherlands,* 604–609.

Mustafa, S., Bahar, A., Aziz, Z. A., & Suratman, S. (2016). Modelling contaminant transport for pumping wells in riverbank filtration systems. *Journal of Environmental Management,* 165, 159-166, http://dx.doi.org/10.1016/j.jenvman.2015.09.026.

Olayiwola, R. O., Jimoh, O. R., Yusuf, A., & Abubakar, S. (2013). A mathematical study of contaminant transport with first order decay and time-dependent source concentration in an aquifer. *Universal Journal of Mathematics,* 1(2), 112 – 119.

Schwarzenbach, R. P., Egli, T., Hofstetter, T. B., Von Gunten, U., & Wehrli, B. (2010). Global water pollution and human health. *Annual Review of Environment and Resources.* 35(1), 109–136.

Selamat, R., Abustan, I., Arshad, M. R., & Kamal, N. H. M. (2019). Removal of *Escherichia Coli* Using Low-Frequency Electromagnetic Field in Riverbank Filtration. *Water and Wastewater Treatment,* 1-19, *DOI:http://dx.doi.org/10.5772/intechopen.85296*.

Shende, S., & Chau, K. (2019). Attenuation of pathogen shock load: Analytical analysis of infiltration gallery in riverbank filtration during stream stage rise. *Journal of Hazardous, Toxic, and Radioactive Waste,* 23(4), 04019009- 8, DOI: 10.1061/(ASCE)HZ.2153-5515.0000445.

Singh, V. P. (2017). "Challenges in meeting water security and resilience." Water Int. 42 (4): 349–359, https://doi.org/10.1080/02508060.2017.1327234.

U.S. Environmental Protection Agency (2009). Water on Tap: What you need to know. Technical report.

WHO (2006). Guidelines for drinking-water quality, third edition, incorporating first addendum. Groundwater protection zones, *World Health Organization.*

WHO (2017). Potable reuse: Guidance for producing safe drinking-water*World Health Organization.*