A COMPUTATIONAL ANALYSIS OF SINGULARITY POTENTIAL INDUCED CHARACTERISATION OF HIGHER MOMENTS ON INSURANCE CLAIM SIZE IN A LOSS EVENT

[1]*OGUNGBENLE, GBENGA MICHAEL; [2]KUSA, NANFA DANJUMA & [3]ADEOYE, OLUSEYI VINCENT

[1]Department of Actuarial Science, Faculty of Management Sciences, University of Jos
[2]Department of Business Administration, Faculty of Management Sciences, University of Jos
[3]Department of Banking & Finance, Faculty of Management Sciences, University of Jos
[1]Correspondence: gbengarising@gmail.com

Abstract

The paper theoretically examines the behavior of claim size function with a deductible structure in a loss event under the potential of singularity kernel. It is a separate and further extension of the work of Ogungbenle et al. 2020, where singularity function was initiated to study differential equations which govern actuarial problems in actuarial risk theory. The effect of the structural properties of the singularity kernel on the expected loss as it relates to general insurance business have been studied. Consequently, the asymptotic claim that singularity potential at any point on the real line domain where the integral over the extended real line of the product of a function and dirac-delta always produces the functional value of the function at that material point form the basis of our arguments throughout. We have also constructed a novel method of modelling the characteristic function of a complex random risk as part of our contributions and findings. The results of our investigation show that the amount of claim size and its higher moments in a loss event is a function of the coverage modification parameter and the expected value of the individual random risk.

Keywords: Actuarial present value, claim size, singularity potential, moments.

1. Introduction to Singularity Functions

This paper aims to theoretically investigate the behavior of actuarial functions under the potential of singularity kernel. It is a separate and further extension of the work of Ogungbenle et al, (2020) where singularity function was initiated to model differential equations which govern actuarial problems in risk theory. In (Dirac,1930; Pazman & Pronzato,1996; Saichev & Woyczynski,1997; Onural, 2006 and Salansnich, 2014), the discontinuous Heaviside function of the first kind at the point $x=0$ was discovered which is usually applied in the analysis of electrical circuit quantum physics, and statistical physics. The quantum distribution of electrons in metallic object was described where the Fermi- Dirac probability density function for the distribution of electron was formulated as follows

$$F_{\chi}(x)=\frac{1}{e^{\chi x}+1}=\int_{-\infty}^{x} f_{\chi}(s)\,ds \tag{1}$$

$$\lim_{\chi\to\infty} F_{\chi}(x)=\lim_{\chi\to\infty}\frac{1}{e^{\chi x}+1}=\begin{cases}0, if, x>0\\ 1, if, x<0\end{cases} \tag{2}$$

and hence the probability density function is

$$F_{\chi}'(x)=f_{\chi}(x)=-\frac{\chi e^{\chi x}}{\left(e^{Xx}+1\right)^{2}}=-\chi e^{\chi x}\left(F_{\chi}(x)\right)^{2} \tag{3}$$

where, $X=(\in-\psi)$, represents the deviation of the electron's energy $\in$ from the chemical potential ψ. if T is the absolute temperature and K_χ Boltzman constant, then the inverse of the absolute temperature,

$$J(T)=\frac{1}{K_\chi T} \tag{4}$$

We investigate the limiting value of $f_\chi(x)$ and $F_\chi(x)$ as χ approaches infinity

$$\lim_{\chi\to\infty} f_\chi(x)=\lim_{\chi\to\infty}\left[\frac{x\chi e^{x\chi}+e^{x\chi}}{2x\left(e^{x\chi}+1\right)e^{x\chi}}\right]=\lim_{\chi\to\infty}\left[\frac{x\chi+1}{2x\left(e^{x\chi}+1\right)}\right]=\lim_{\chi\to\infty}\left[\frac{1}{2xe^{x\chi}}\right] \tag{5}$$

$$\lim_{\chi\to\infty} f_\chi(x)=\left[\frac{1}{2xe^{x\chi}}|\chi=\infty\right]=\begin{cases}0, if, x>0\\ \infty, if, x<0\end{cases} \tag{6}$$

From the computation of the limiting value above there is hardly any risk function which can satisfy the condition of the limit function and the integral simultaneously. This issue motivated (Dirac,1930 and Zhang,2018) to carry out an independent work in deriving the density of extreme and localized charge and reformulated the problem created by the above two condition so that probability density function can meet the conditions above simultaneously. In view of (Pazman & Pronzato,1996; Kanwal,1998; Khuri,2004; Mohammed,2011; Ogungbenle et al, 2020), the problem that no single risk function can satisfy both conditions can be avoided by redefining the limit function and integral stated in equations (7) and (8) below

$$\int_{-\infty}^{\infty}\delta(s)ds=\lim_{\eta\to 0^+}\int_{-\infty}^{\infty}\delta(\eta,s)ds=\lim_{\eta\to 0^+}\int_{-b}^{b}\delta(\eta,s)ds, b\to\infty \tag{7}$$

$$\lim_{\eta\to 0}\delta(\eta,s)=\begin{cases}\infty, if, x=0\\ 0, if, x\neq 0\end{cases} \tag{8}$$

$$\int_{-\infty}^{\infty}\delta(s)ds=1 \tag{9}$$

We see from (Salansnich,2014 and Zhang, 2018) that certain probability density functions such as Gaus(x) can meet the two conditions above. The Gausian function could be examined in the limit to check the authors' claim.

$$G(x,\eta)=\frac{e^{-x^2\eta^{-2}}}{\eta\pi} \tag{10}$$

$$\lim_{\eta\to 0^+} G(x,\eta)=\lim_{\eta\to 0^+}\frac{e^{-x^2\eta^{-2}}}{\eta\pi}\cong\lim_{\eta\to 0^+}\left[\frac{1}{\eta\pi}+\frac{x^2}{\eta^3\pi}\right]=\begin{cases}\infty, x=0\\ 0, x>0\\ 0, x<0\end{cases} \tag{11}$$

implying that the limit $\lim_{\eta\to 0\uparrow} G(x,\eta)$ is actually not in indeterminate form. The probability is defined on the real line and the integral of probability density function on the real line is 1, we have,

$$\int_{-\infty}^{\infty}G(x,\eta)dx=\int_{-\infty}^{\infty}\frac{e^{-x^2\eta^{-2}}}{\eta\pi}dx=\int_{0}^{\infty}\frac{e^{-x^2\eta^{-2}}}{\eta\pi}dx=1 \tag{12}$$

Theorem: Suppose y represents a random risk and d is the deductible then under the singularity potential, the following condition holds

$$\frac{\mathrm{dln}\ \delta\left(\mathrm{y}-d\right)}{\mathrm{dy}}=-\frac{1}{\mathrm{y}}\ \text{and}\ \int_{-\infty}^{\infty}\delta'\left(\mathrm{y}-d\right)\mathrm{y}^{(2\mathrm{n})}\mathrm{dy}=-\left(2\mathrm{n}\right)d^{(2\mathrm{n}\text{-}1)}$$

Proof

$$\int_{-\infty}^{\infty}\delta'\left(\mathrm{y}-d\right)\mathrm{ydy}=y\delta\left(\mathrm{y}-d\right)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\delta\left(\mathrm{y}-d\right)\mathrm{dy}=-\int_{-\infty}^{\infty}\delta\left(\mathrm{y}-d\right)\mathrm{dy}=1 \quad (13)$$

$$\int_{-\infty}^{\infty}\delta'\left(\mathrm{y}-d\right)\mathrm{ydy}=-\int_{-\infty}^{\infty}\delta\left(\mathrm{y}-d\right)\mathrm{dy} \quad (14)$$

$$\delta'\left(\mathrm{y}-d\right)\mathrm{y}=-\delta\left(\mathrm{y}-d\right)\Rightarrow\delta'\left(\mathrm{y}-d\right)=\frac{-\delta\left(\mathrm{y}-d\right)}{\mathrm{y}}\ \text{and}\ \delta'\left(\mathrm{y}\right)\mathrm{y}=-\delta\left(\mathrm{y}\right),\text{when},\mathrm{d}=0$$

$$-\frac{\delta'\left(\mathrm{y}-d\right)}{\delta\left(\mathrm{y}-d\right)}=\mathrm{y}^{-1}\ \text{implying that}\ \frac{\mathrm{dln}\ \delta\left(\mathrm{y}-d\right)}{\mathrm{dy}}=-\frac{1}{\mathrm{y}} \quad (15)$$

$$\int_{-\infty}^{\infty}\delta'\left(\mathrm{y}-d\right)\mathrm{y}^2\mathrm{dy}=\mathrm{y}^2\delta\left(\mathrm{y}-d\right)\Big|_{-\infty}^{\infty}-2\int_{-\infty}^{\infty}\mathrm{y}\delta\left(\mathrm{y}-d\right)\mathrm{dy}=-2\int_{-\infty}^{\infty}\mathrm{y}\delta\left(\mathrm{y}-d\right)\mathrm{dy}=-2d \quad (16)$$

$$\int_{-\infty}^{\infty}\delta'\left(\mathrm{y}-d\right)\mathrm{y}^3\mathrm{dy}=\mathrm{y}^3\delta\left(\mathrm{y}-d\right)\Big|_{-\infty}^{\infty}-3\int_{-\infty}^{\infty}\mathrm{y}^2\delta\left(\mathrm{y}-d\right)\mathrm{dy}=-3\int_{-\infty}^{\infty}\mathrm{y}^2\delta\left(\mathrm{y}-d\right)\mathrm{dy}=3\mathrm{d}^2 \quad (17)$$

$$\int_{-\infty}^{\infty}\delta'\left(\mathrm{y}-d\right)\mathrm{y}^4\mathrm{dy}=\mathrm{y}^4\delta\left(\mathrm{y}-d\right)\Big|_{-\infty}^{\infty}-4\int_{-\infty}^{\infty}\mathrm{y}^3\delta\left(\mathrm{y}-d\right)\mathrm{dy}=-4\int_{-\infty}^{\infty}\mathrm{y}^3\delta\left(\mathrm{y}-d\right)\mathrm{dy}=-4\mathrm{d}^3 \quad (18)$$

$$\int_{-\infty}^{\infty}\delta'\left(\mathrm{y}-d\right)\mathrm{y}^{(2\mathrm{n})}\mathrm{dy}=\mathrm{y}^{(2\mathrm{n})}\delta\left(\mathrm{y}-d\right)\Big|_{-\infty}^{\infty}-2\mathrm{n}\int_{-\infty}^{\infty}\mathrm{y}^{(2\mathrm{n}\text{-}1)}\delta\left(\mathrm{y}-d\right)\mathrm{dy}= \quad (19)$$

$$-\left(2\mathrm{n}\right)\int_{-\infty}^{\infty}\mathrm{y}^{(2\mathrm{n}\text{-}1)}\delta\left(\mathrm{y}-d\right)\mathrm{dy}$$

$$\int_{-\infty}^{\infty}\delta'\left(\mathrm{y}-d\right)\mathrm{y}^{(2\mathrm{n})}\mathrm{dy}=-\left(2\mathrm{n}\right)d^{(2\mathrm{n}\text{-}1)} \quad (20)$$

$$\int_{-\infty}^{\infty}\delta'\left(\mathrm{y}-d\right)\mathrm{y}^{(2\mathrm{n}-1)}\mathrm{dy}=\mathrm{y}^{(2\mathrm{n}-1)}\delta\left(\mathrm{y}-d\right)\Big|_{-\infty}^{\infty}-\left(2\mathrm{n}-1\right)\int_{-\infty}^{\infty}\mathrm{y}^{(2\mathrm{n}\text{-}2)}\delta\left(\mathrm{y}-d\right)\mathrm{dy}= \quad (21)$$

$$-\left(2\mathrm{n}-1\right)d^{(2\mathrm{n}-2)}$$

2 The Mathematical Theory of Dirac-Delta Function

Applications of ordinary differential equations are currently employed in modeling insurance cash flows and change in all discipline of actuarial science. This application has become an important technique of risk analysis essentially in general insurance business. Most problem in actuarial risk literature concerns the development of model for general insurance and casually. It is on this basis we use singularity function to investigate the behavior of actuarial function in a new dimension and then derive model in actuarial statistic and casually.

The second order differential equation

$$\alpha_1\frac{d^2y}{dy^2}+\alpha_2\frac{dy}{dy}+\alpha_3 y=g\left(s\right) \quad (22)$$

is deep rooted in many branches of actuarial discipline, especially in financial engineering where it has been used to analyse the term structure of interest rates by setting the forcing function

$g(s)=0$ and further assuming that the homogenous differential equation $\alpha_1 \frac{d^2y}{dy^2}+\alpha_2 \frac{dy}{dy}+\alpha_3 y=0$ has equal real roots with constant co-efficient $\alpha_{i,} i=1,2,3.$ Following (Balcer & Sahin, 1979; Saichev & Woyczynski, 1997; Kanwal,1998; Khuri, 2004; Onural, 2006; Sastry, 2009; Mohammed, 2011and Ogungbenle et al., 2020), a core application of integral transform occurs in dealing with ordinary differential equation with discontinuous forcing function structure particularly in the analysis of step function and in engineering, where the characteristic equations defining the behavior of an electric circuit in the complex frequency is associated with linear combinations of exponentially scaled and time-shifted damped sine wave in the time domain. Furthermore, the integral transform is used to map a function from its original function space into another function space through integration such that the behaviour of the original function could be more conveniently characterized than in the original function space. Consequently, the transformed function is then mapped back to the original function space through the use of inverse transform.

In Ogungbenle et al., (2020), the dirac-delta function was obtained through the second order differential equation $\alpha_1 y''+\alpha_2 y'+\alpha_3 y=g(s)$ where $g(s)$ above is a measure of forcing term and the total area under the curve $\int_{-\infty}^{\infty} g(s)ds=\lim_{a\to\infty}\int_{-a}^{a} g(s)ds,$ is the impulsive force. By definition, the forcing term is

$$g(s)=\begin{cases} \infty, s\in[s_0-\eta, s_0+\eta] \\ 0, s<s_0-\eta; OR, s>s_0+\eta \end{cases} \tag{23}$$

Here $\eta<\varepsilon$ where ε is a small positive number. Hence, the impulsive force is

$$h(s)=\int_{s_0-\eta}^{s_{0+\eta}} g(s)ds \tag{24}$$

Since,

$$g(s)=0, s<s_0-\eta, or, s>s_0+\eta \text{, then } h(s)=\int_{-\infty}^{\infty} g(s)ds \tag{25}$$

By the mean value theorem for integral,

$$h(s)=\int_{s_0-\eta}^{s_0+\eta} g(s)ds=2\eta g(\bar{s}) \tag{26}$$

$\bar{s}\in[s_0-\eta, s_0+\eta], and, \bar{s}\to s_0 as \eta\to 0$

(Kanwal,1998 and Salansnich,2014) assume $\eta\to 0$ in the interval and then define the function

$$\lambda_r(s)=g(s)=\begin{cases} \frac{1}{2\eta,}; s\in[-\eta,\eta] \\ 0, s<-\eta, or, s>\eta \end{cases} \tag{27}$$

$$h(s)=\int_{-\eta}^{\eta} g(s)ds=\int_{-\eta}^{\eta}\frac{1}{2\eta}ds=\frac{1}{2\eta}\times 2\eta=1 \tag{28}$$

Now $\lim_{\eta\to 0}\lambda_r(s)=\lim_{\eta\to 0}\frac{1}{2\eta}=\frac{0}{2}=0, \eta\neq 0$ using the L'Hopital's rule $\lim_{\eta\to 0} h(s)=1$ that is the Limiting value of total impulsive force is 1. The integral value 1 and the limiting value 0 both define the value of dirac-delta function δ which has a value 1 when $s=0$ and 0 if otherwise.

$$\lim_{\eta\to 0}\int_{-\infty}^{\infty}\lambda_r\left(s-s_0\right)g\left(s\right)ds=\lim_{\eta\to 0,a\to\infty}\int_{-a}^{a}\lambda_r\left(s-s_0\right)s\left(s\right)ds=\int_{-\infty}^{\infty}\delta\left(s-s_0\right)g\left(s\right)ds \quad (29)$$

$$2\eta\lim_{a\to\infty}\left[\int_{-a}^{a}\lambda_r\left(s-s_0\right)g\left(s\right)ds\right]=2\eta\int_{-\infty}^{\infty}\delta\left(s-s_0\right)g\left(s\right)ds=\int_{s_0-\eta}^{s_0+\eta}g\left(s\right)ds\approx 2\eta g\left(\bar{s}\right)$$

(30)

$if,\int_{-\infty}^{\infty}\delta\left(s-s_0\right)g\left(s\right)ds\approx g\left(\bar{s}\right),\ \delta\left(s-s_0\right)$ is the kernel of the integral transform describing the dimensions of a rectangular parralleliped of length 2η and height $\frac{1}{2\eta}$ and centered at s_0 so that its area will be 1. The function $\delta(s-s_0)$ isolates the real value of $g\left(s\right)$ at some prescribed point s_0 but as stated above that $\bar{s}\to s_0$ as $\eta\to 0$ hence

$$\int_{-\infty}^{\infty}\delta\left(s-\bar{s}\right)g\left(s\right)ds\approx g\left(s_0\right) \quad (31)$$

$$\delta\left(s-\bar{s}\right)=\begin{matrix}0, if, s\neq\bar{s}\\ \infty, if, s=\bar{s}\end{matrix} \quad (32)$$

Invoking the normalizing condition, $\delta\left(s-\zeta\right)=\delta\left(\zeta-s\right)$, then

$$\int_{-\infty}^{\infty}\delta\left(s-\bar{s}\right)ds=\int_{-\infty}^{\infty}\delta\left(s-\bar{s}\right)ds\Rightarrow\int_{-\infty}^{\infty}\left[2\bar{s}\delta\left(s^2-\left(\bar{s}\right)^2\right)-\delta\left(s+\bar{s}\right)\right]ds=1 \quad (33)$$

Letting $t=\left(s-\bar{s}\right),\int_{-\infty}^{\infty}\delta(t)dt=1$ (34)

3 Decomposition OF Risk Measure

The authors in (Saichev & Woyczynski, 1997) investigated marginal decomposable risk measure $\theta(Z)$ and observed that the analysis of risk measure begins by pooling insured peril Y_i together that is $Z=\sum_i Y_i$ and a risk measure $\theta(Z)$ expressing some component of the pool. The components are usually the insurance lines of business. The composition of the component units defines a function $R(Y)$ such that $\theta(Z)=\sum_i R(Y_i)$ where Z is the total risk insured and is assumed to be continuous. Whenever, the risk measure for an insurer is obtained as a sum function of related business unit risk measures, then the original risk measure is the decomposition of risk measure to the business units. The variance of Y_i will not total up to the variance of Z except Y_i are all independent random variables thus the allocation of variance as a function of the variances of the component units cannot be decomposed unless there is independence. It can then be inferred that if each line of insurance business is assigned its marginal impact, then the decomposition of the risk measure $\theta(Z)$ is marginal where the impact described a small incremental change or the whole insurance lines of insurance business.

3.1 Density Function Using Dirac-Delta Kernel

This section lays the foundation for the application of the actuarial statistics and singularity potential, part of which is recalled from Ogungbenle, et al. (2020) to form the basis of our investigation.

In view of (Jack & Ormiston,1999; Venter *et al.*, 2006; Tse,2009) and Ogungbenle, et al. (2020), we recall the followings to enable us explain the use of dirac-delta function. We let $X_{j,}\, j = 1,2,3,...m$ be the size of the jth random risk with frequency

$f_{X_j}(z), 1, 2, 3, ,...m$ and such that if $\sum_{j=1}^{m} P_j = 1$. We let

$$f_{X_j}(z_1) = P_1\delta(z_1 - z_1^{\bullet}) + P_2\delta(z_1 - z_2^{\bullet}) + ... + P_m\delta(z_1 - z_m^{\bullet}) = \sum_{j=1}^{m} P_j\delta(z_1 - z_j^{\bullet}) \quad (35)$$

Where z_j^* are the function values of X_1

Hence, we can define $f_{X_1,X_2}(z_{1,} z_2) = \sum_{j=1}^{n}\sum_{k=1}^{m} P_{jk}\delta(z_1 - z_j^*)\delta(z_2 - z_k^*)$ (36)

where z_k^* are the functional values of random risk X_2

$$f_X(x) = f_{X_1,X_2}(z_1, z_2) = \frac{\partial^2 F_{X_1,X_2}(z_1, z_2)}{\partial z_1 \partial z_2} = \sum_{j=1}^{n}\sum_{k=1}^{m} P_{jk}\delta(z_1 - z_j^*)\delta(z_2 - z_k^*) \quad (37)$$

$$f_1(z_1) = \sum_{x_2 \in \Omega_2} \frac{\partial^2 F_{X_1,X_2}(z_1, z_2)}{\partial x_1 \delta x_2} = \sum_{x_2 \in \Omega_2} f_{X_1,X_2}(z_1, z_2)$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{m} P_{jk}\delta(z_1 - z_j^*)\delta(z_2 - z_k^*) \quad (38)$$

$$f_2(z_2) = \sum_{x_1 \in \Omega_1} f_{X_{11},X_2}(z_1, z_2) = \sum_{k=1}^{n}\sum_{j=1}^{m} P_{kj}\delta(z_1 - z_j^*)\delta(z_2 - z_k^*) \quad (39)$$

$$\sum_{z_1 \in \Omega_1} f_1(z_1) = \sum_{z_1 \in \Omega_1}\sum_{z_2 \in \Omega_2} f_{X_{11}X_2}(z_1, z_2) \quad (40)$$

$$\sum_{z_2 \in \Omega_2} f_2(z_2) = \sum_{z_2 \in \Omega_2}\sum_{z_1 \in \Omega_1} f_{X_{11},X_2}(z_1, z_2) \quad (41)$$

$$f_{X_2|X_1}(z_2|z_1) = \frac{f_{X_1,X_2}(z_1, z_2)}{f_1(z_1)} \quad (42)$$

We recall from Ogungbenle et al. (2020), that this is a function of z_2 but z_1 is arbitrary value of $f_1(z_1) > 0$

$$\sum_{z_2} f_{X_2|X_1}(z_2|z_1) = \frac{\sum_{z_2 \in \Omega_2} f_{X_{11}X_2}(z_1, z_2)}{f_1(z_1)} \quad (43)$$

$$\sum_{X_1} f_{X_1|X_2}(z_1|z_2) = \frac{\sum_{X_{1\in} \in \Omega_1} f_{X_{11}X_2}(z_1, z_2)}{f_2(z_2)} \quad (44)$$

$$f_{X_1|X_2}(z_1|z_2) = \frac{f_{X_1,X_2}(z_1, z_2)}{f_2(z_2)} \quad (45)$$

$$f_{X_2|X_1}(z_2|z_1) = \frac{f_{X_1,X_2}(z_1, z_2)}{f_1(z_1)} = \frac{\Pr(X_1 = z_1 . X_2 = z_2)}{\Pr(X_1 = z_1)} \quad (46)$$

$$f_{X_2|X_2}\left(z_2|z_1\right)=\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}p_{jk}\delta\left(z_1-z_j^{*}\right)\delta\left(z_2-z_k^{*}\right)}{\sum_{j=1}^{m}p_j\delta\left(z_1-z_j^{*}\right)} \quad (47)$$

Thus following (Pazman & Pronzato, 1996; Kanwal, 1998; Khuri, 2004; Tse, 2009; Mohammed, 2011 and Ogungbenle et al., 2020), the conditional density of the expected loss is defined. In the continuous case, we have

$$f_1\left(z_1\right)=\int_{-\infty}^{\infty}f_{X_1X_2}\left(z_1,z_2\right)dz_2,\ f_2\left(z_2\right)=\int_{-\infty}^{\infty}f_{X_1X_2}\left(z_1,z_2\right)dz_1 \quad (48)$$

$$\int_{-\infty}^{\infty}f_1\left(z_1\right)dz=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{X_1X_2}\left(z_1,z_2\right)dz_1dz_2$$

$$f_{X_1|X_2}\left(z_1|z_2\right)=\frac{f_{X_1,X_2}\left(z_1,z_2\right)}{f_2\left(z_2\right)} \quad (49)$$

The joint density function,

$$f_{X_1,X_2}\left(z_1,z_2\right)=\iint f\left(z_1,z_2\right)\delta\left(z-z_1,z-z_2\right)dz_1dz_2 \quad (50)$$

$$f_{X_1|X_2}\left(z_1|z_2\right)=\frac{f_{X_1,X_2}\left(z_1,z_2\right)}{f_2\left(z_2\right)} \quad (51)$$

$$f_{X_1|X_2}\left(z_1|z_2\right)=\frac{\iint f\left(z_1,z_2\right)\delta\left(z_1-z^{*},z_2-z^{*}\right)dz_1dz_2}{\int f_{X_1X_2}\left(z_1,z_2\right)\delta\left(z_2-z^{*}\right)dz_1} \quad (52)$$

4 The Characteristic Function Of A Complex Random Risk

Let X be a random risk with density $f_X\left(x\right)$ and $i=\sqrt{-1}$, then the characteristic function of X defined by $\theta_X\left(s\right)=E\left(e^{isx}\right)$

$$\theta_x\left(s\right)=\sum_{X\in\Omega_x}\left[\left(\cos sx\right)f_X\left(x\right)+i\left(\sin x\right)f_X\left(x\right)\right] \quad (53)$$

$$\theta_x\left(s\right)=\sum_{x\in\Omega_x}\left[\left(\cos sx\right)f_X\left(x\right)\right]+i\sum_{x\in\Omega_x}\left[\left(\sin sx\right)f_X\left(x\right)\right] \quad (54)$$

and in the continuous sense, we have

$$\theta_x\left(s\right)=E\left(e^{isx}\right)=\int_{-\infty}^{\infty}\left[\left(\cos sx\right)f_x\left(x\right)+i\left(\sin sx\right)f_x\left(x\right)\right]dx \quad (55)$$

$$\theta_x\left(s\right)=\int_{-\infty}^{\infty}\left[\left(\cos sx\right)f_x\left(x\right)\right]dx+\int_{-\infty}^{\infty}\left[i\left(\sin sx\right)f_x\left(x\right)\right]dx \quad (56)$$

$$\theta_x\left(s\right)=E\left(e^{isx}\right)=$$

$$\int_{-\infty}^{\infty}\left[\left(\cos sx\right)\sum_{j=1}^{m}P_j\delta\left(X-x_j^{*}\right)+i\left(\sin sx\right)\sum_{j=1}^{m}P_j\delta\left(X-x_j^{*}\right)\right] \quad (57)$$

$$\theta_x\left(s\right)=\int_{-\infty}^{\infty}\left[\left(\cos sx\right)\sum_{j=1}^{m}P_j\delta\left(X-x_j^{*}\right)+i\left(\sin sx\right)\sum_{j=1}^{m}P_j\delta\left(X-x_j^{*}\right)\right]dx \quad (58)$$

$$\theta_x(s) = \sum_{j=1}^{m} \int_{-\infty}^{\infty} (\cos sx) P_j \delta\left(X - x_j^*\right) dx + i \sum_{j=1}^{m} \int_{-\infty}^{\infty} (\sin sx) P_j \delta\left(X - x_j^*\right) dx \tag{59}$$

$$\theta_x(s) = \sum_{j=1}^{m} P_j \int_{-\infty}^{\infty} (\cos sx) \delta\left(X - x_j^*\right) dx + i \sum_{j=1}^{m} P_j \int_{-\infty}^{\infty} (\sin sx) \delta\left(X - x_j^*\right) \tag{60}$$

$$\theta_x(s) = \sum_{j=1}^{m} P_j \cos sx_j^* + i \sum_{j=1}^{m} P_j \sin sx_j^* \tag{61}$$

Let $f(x)$ be a continuously smooth and integrable function of a random risk, then recalling from equation (31),

$$f(s_0) = \int_{-\infty}^{\infty} \delta\left(s - \bar{s}\right) f(s) ds \tag{62}$$

$$f(s_0) = \int_{-\infty}^{\infty} \delta\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iV(s-t)} dV\right) f(s) ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iVt} \int_{-\infty}^{\infty} e^{iVs} dV f(s) ds \tag{63}$$

$$\delta(s - t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iV(s-t)} dV \tag{64}$$

Thus if $t = 0$, then we have

$$\delta(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iVs} dV \tag{65}$$

The fast Fourier transform FFT of $f(\cdot)$ is defined as

$$FFT(V) = \int_{-\infty}^{\infty} e^{iVs} f(s) ds \tag{66}$$

$$f(t) = \int_{-\infty}^{\infty} e^{-iVt} FFT(V) dV \tag{67}$$

Requiring the probability density is to be evaluated for a defined number of arguments, the fast Fourier formula is applicable so as to enhance the speed of the convergent integral. The function $\theta_x(s)$ wholly specifies the distribution of random variable X to the extent that when $\theta_x(s) = \theta_Y(s)$ X and Y will be identically distributed but if X and Y are independent random variables $\theta_{X+Y}(s) = \theta_Y(s)\theta_X(s), s \in \mathbb{R}^+$. The characteristic function describes the Fourier transform of the probability density function of a random actuarial risk such that $FFT(V) = \int_{-\infty}^{\infty} e^{iVs} f(s) ds.$ The function $f(s)$ can be used to obtain the final pay–off to a unit linked insurance which is maturing at time s

5 The Method Of Obtaining Moments Of Actuarial Risk By Dirac-Delta Kernel

In (Jack & Ormiston,1999; Venter, Mojor & Kreps, 2006; Sakthivel & Rajitha,2017and Lui & Wang , 2017), the policy could be written such that the insured would be responsible for part of the risk by introducing policy excess (deductible) in the policy terms and conditions of the coverage. However, drawing inference from (Schlesinger,1981; Schlesinger,1985; Thogersen, 2016 and Frees, Lee & Yang,2016), it is possible that there may be a reduction in the average amount and variability in amount paid out by the underwriter such that the probability that the underwriter will experience a high claim pay-out on block of claims will damp out.

Assume that the loss incurred by the policy holder is X but Y defines the part of the loss incurred by the insurance firm. The payment per loss random variable describes the losses over which a payment has been made as well as losses lower than the deductible d hence $Total\ loss(X) = Insurer's\ paid\ claim(Y) + insured's\ loss(U)$ and consequently the amount of claim size in the loss event is defined as follows

$$X_L = Y_d = \begin{cases} 0, X \le d \\ X - d, X > d \end{cases}$$

$$X_+ = \begin{cases} 0, X \le 0 \\ X, X > 0 \end{cases} \tag{68}$$

and $E(U_d) = d(S_X(d)) + \int_0^d y f_X(y)\,dy$ (68a)

$X_L = (X - d)_+$, where

$$E(Y_d) = E(X - d)_+ = \int_0^\infty \Pr(X > d + x)\,dx$$

(69)

$$\Pr(X_L = 0) = F_X(d)$$

(70)

X_L is a censored and random shifted variable by reason of the fact that claim values lower than d have not been ignored and all losses have been shifted away by d

In view of (Mojor & Kreps, 2006 and Tse, 2009), X_L has a probability mass point at zero of $F_X(d)$

$$f_{X_L}(x) = f_X(x + d),\ for, x > 0 \tag{71}$$

$$E(X_L) = E(X - d)_+ - d(S_X(d)) - \int_0^d y f_X(y)\,dy \tag{71a}$$

$$E(X_L) = \int_{-\infty}^{\infty} (X - d)\,dF_{X_L}(x) \tag{72}$$

$$E(X_L) = \int_0^\infty (X - d)\,dF_{X_L}(x) = E(X - d)$$

$$= \int_0^\infty \Pr(X > d + x)\,dF_{X_L}(x) \tag{73}$$

$$\theta = d + x \Rightarrow dx = d\theta$$

We can determine the expected value of loss as, letting $E\left(X_L\right)=\int_d^{\infty}\Pr\left(X>\theta\right)f_{X_L}\left(x\right)d\theta$ by the requirement of deductible. It is observed in (Schlesinger,1985; Venter, Mojor & Kreps, 2006; Thogersen, 2016; Lui and Wang, 2017) that the indemnity function

$Ind\left(L\right)=E\left(X-d\right)_+=\max\left(\left(X-d\right),0\right)$

and $\Pr\left(\left(X-d\right)_+>x\right)=\Pr\left(X>x+d\right)$ (74)

$$E\left(X_L\right)=\int_d^{\infty}\left(X-d\right)\sum_{j=1}^{m}P_j\delta\left(X-x_j^{*}\right)dx \quad (75)$$

$$E\left(X_L\right)=\sum_{j=1}^{m}P\int_d^{\infty}\left(X-d\right)\delta\left(X-x_j^{*}\right)dx \quad (76)$$

$$E\left(X_L\right)=\sum_{j=1}^{m}P_j\left(X_j^{*}-d\right)=\sum_{j=1}^{m}P_jX_j^{*}-\sum_{j=1}^{m}dP_j \quad (77)$$

$$E\left(X_L\right)=\sum_{j=1}^{m}P_jx_j^{*}-d\sum_{j=1}^{m}P_{j,}\ \text{Recall that } \sum_{j=1}^{m}P_j=1 \quad (78)$$

$$E\left(X_L\right)=\sum_{j=1}^{m}P_jx_j^{*}-d=E\left(x_j^{*}\right)-d=\mu_j^{\bullet}-d \quad (79)$$

Thus, in view of (Venter, et al., 2006; Gomez-Deniz 2016; Valecky 2016; Garrido, et al., 2016; and Sakthivel & Rajitha, 2017), the above represents the expected claim liability under the deductible policy contract that the insurer is liable to pay.

The insurer's loss on the contract is both non-negative and potentially large. In order to model claim of large sizes using insurance data, the loss probability densities using frequency-severity conditions usually give room for high value distribution normally skewed to the right and fat-tailed in practical setting. In actuarial casualty, the tail of the fat tailed-distributions are usually not exponentially bounded, that is, for any $x>0$, the density $f_X\left(x\right)$will not be of the form $\Pr\left(X>x\right)\le\alpha e^{-\beta x}$

Furthermore, we infer in (Schlesinger,1981; Garrido, Genest & Schulz,2016; Park; Kim & Ahn,2018; Woodard & Yi, 2018), actuaries make use of loss probability models to estimate the value of monetary loss of an insurance claim since the ultimate goal of the underwriter is to obtain the total value of claims so as to find a convenient value for both premium and reserves. Actuarial loss distribution shows the probability of a severity of a defined amount and the probability of a loss being higher or falling below a defined loss size. The loss distribution could be applied to compute the expected proportion of the aggregate severities in excess of a certain Naira-threshold or the expected losses in excess of the deductible amount. With a deductible condition, the underwriter grants cover for the less predictable severity with a bigger random component structure. Consequently, the underwriter may wish to include a particular incremental risk margin. Although the estimation of risk margins is not within the scope of this paper, it is assumed for explanation purposes that the underwriter could add an additional risk margin κ% of excess losses. The second moment is defined as follows.

$$E\left(X^2{}_L\right)=\int_{-\infty}^{\infty}\left(X-d\right)^2 f_{X_L}\left(x\right)dx=\int_{-\infty}^{\infty}\left(X-d\right)^2 dF_{X_L}\left(x\right) \quad (80)$$

Since density is only defined on the real line, we integrate from zero to infinity

$$E\left(X^2{}_L\right)=\int_0^\infty \left(X-d\right)^2 dF_{X_L}\left(x\right) \tag{81}$$

By the definition of deductible, we integrate within $\left(d,\infty\right)$

$$E\left(X^2{}_L\right)=\int_d^\infty \left(X-d\right)^2 \sum_{j=1}^m P_j\delta\left(X-x_j^*\right)dx \tag{82}$$

$$E\left(X^2{}_L\right)=\sum_{j=1}^m P_j\int_d^\infty \left(X-d\right)^2\delta\left(X-x_j^*\right)dx \tag{83}$$

$$E\left(X^2{}_L\right)=\sum_{j=1}^m P_j\left(x_j^*-d\right)^2=\sum_{j=1}^m P_j\left(x_j^*\right)^2-2d\sum_{j=1}^m P_jx_j^*+\sum_{j=1}^m P_jd^2 \tag{84}$$

$$E\left(X^2{}_L\right)=\sum_{j=1}^m P_j\left(x_j^*\right)^2-2d\sum_{j=1}^m P_jx_j^*+d^2 \tag{85}$$

$$E\left(X^2{}_L\right)=\left(E\left(x_j^*\right)\right)^2-2dE\left(x_j^*\right)+d^2 \tag{86}$$

$$E\left(X^2{}_L\right)=\left(\mu_j^\bullet\right)^2-2d\mu_j^\bullet+d^2 \tag{87}$$

To illustrate this basic concept therefore, we assume that the loss is exponentially distributed with mean loss ₦M and that the insurer will indemnify the value of the loss in excess of deductible pricing ₦b . it is possible to obtain the variance of the value identified on one claim.

$$X_L=\left(X-b\right)_+=\begin{cases}0, X\le b\\ X-b, X>b\end{cases} \tag{87a}$$

$$E\left(X_L\right)=E\left(X-b\right)_+=\int_b^\infty (x-b)f_X(x)dx=\int_b^\infty S_X(x)dx=Me^{\frac{b}{M}} \tag{87b}$$

$$E\left(\left(X_L\right)^2\right)=\int_b^\infty (x-b)^2 f_X(x)dx=2\times M^2e^{\frac{b}{M}} \tag{87c}$$

$$V\left(X_L\right)=\int_b^\infty (x-b)^2 f_X(x)dx-\left[\int_b^\infty (x-b)f_X(x)dx\right]^2 \tag{87d}$$

$$V\left(X_L\right)=2M^2e^{\frac{b}{M}}-M^2e^{\frac{2b}{M}}=M^2e^{\frac{b}{M}}\left(2-e^{\frac{b}{M}}\right)=\frac{1}{2}E\left(\left(X_L\right)^2\right)\left(2-e^{\frac{b}{M}}\right) \tag{87e}$$

as the mean loss M increases, $\left(2-e^{\frac{b}{M}}\right)\to 1$ and $V\left(X_L\right)\to\frac{1}{2}E\left(\left(X_L\right)^2\right)$ (87f)

$$E\left(X^3{}_L\right)=\int_{-\infty}^\infty \left(X-d\right)^3 f_{X_L}(x)dx \tag{88}$$

Again, by the definition of deductible, we integrate for from d

$$E\left(X^3{}_L\right)=\int_d^\infty \left(X-d\right)^3 \sum_{j=1}^m P_j\delta\left(X-x_j^*\right)dx \tag{89}$$

$$E\left(X^3{}_L\right)=\sum_{j=1}^m P_j\int_d^\infty \left(X-d\right)^3\delta\left(X-x_j^*\right)dx \tag{90}$$

$$E\left(X^3{}_L\right)=\sum_{j=1}^{m}P_j\left(x_j{}^*-d\right)^3=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^3-d^3\sum_{j=1}^{m}P_j-3d\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^2 \\ +3d^2\sum_{j=1}^{m}P_j\left(x_j{}^*\right) \tag{91}$$

$$E\left(X^3{}_L\right)=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^3-d^3-3d\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^2+3d^2\sum_{j=1}^{m}P_j\left(x_j{}^*\right) \tag{92}$$

$$E\left(X^3{}_L\right)=\left(E\left(x_j{}^*\right)\right)^3-d^3-3d\left(E\left(x_j{}^*\right)\right)^2+3d^2E\left(x_j{}^*\right) \tag{93}$$

$$E\left(X^3{}_L\right)=\left(\mu_j^\bullet\right)^3-d^3-3d\left(\mu_j^\bullet\right)^2+3d^2\mu_j^\bullet \tag{94}$$

6 Discussion Of Results

It will be necessary to show that the binomial quantity $(x-d)^k$ is well defined.

Recall that the risk exposure x and deductible d are of opposite signs for $d \in \mathbb{R}, x \in \mathbb{R}$

$$\frac{(d-x)^k}{(d-x)^k}=1=(p+q)^k\ ;\ p+q=1$$

Let the real numbers $d>0$ and $x<0$

Assuming that $Y_1, Y_2, Y_3, \ldots, Y_r$ are independent and identically distributed random risk exposure Bernoulli with probability P and let $\lfloor . \rfloor$ be integer function.

Define $p=\dfrac{d}{d+x}, q=1-\dfrac{d}{d+x}=\dfrac{d+x-d}{d+x}=\dfrac{x}{d+x}$

$$\Rightarrow \frac{x}{d+x}-\frac{d}{d+x}=\frac{x-d}{x+d}=-1$$

$E\left\lfloor -\dfrac{(x-d)}{(x-d)}\right\rfloor^{Y_k}=\lfloor(-1)\rfloor^{Y_k}=q-p$, for $k=1,2,3,\ldots,r$

$(q-P)^r=\prod_{k=0}^{r}E\left[(-1)\right]^{Y_k}=E\left[(-1)^{\sum_{k=0}^{r}Y_k}\right]=\sum_{k=0}^{r}\binom{r}{k}(-p)^k(q)^{r-k}$, $x<0$, hence

$$(x-d)^r=\sum_{k=0}^{r}\binom{r}{k}(-d)^r(x)^{r-k}$$

Therefore, computing the kth moment and substituting $\mu_j^\bullet=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)$ if it exists, we have

$$E\left(X^k{}_L\right)=\int_{-\infty}^{\infty}(X-d)^k\,dF_{X_L}(x) \tag{95}$$

$$E\left(X^k{}_L\right)=\int_{d}^{\infty}(X-d)^k\sum_{j=1}^{m}P_j\delta\left(X-x_j{}^*\right)dx \tag{96}$$

$$E\left(X^k{}_L\right)=\sum_{j=1}^{m}P_j\int_{d}^{\infty}(X-d)^k\delta\left(X-x_j{}^*\right)dx \tag{97}$$

$$E\left(X^{k}{}_{L}\right)=\sum_{j=1}^{m}P_{j}\left(x_{j}{}^{*}-d\right)^{k}=\sum_{j=1}^{m}\left[\sum_{r=0}^{k}P_{j}\binom{k}{r}\left(x_{j}{}^{*}\right)^{k-r}\left(-d\right)^{r}\right] \quad (98)$$

$$E\left(X^{k}{}_{L}\right)=\sum_{r=0}^{k}P_{j}\binom{k}{r}\left(x_{j}{}^{*}\right)^{k-r}d^{r}=P_{j}\binom{k}{0}\left(x_{j}{}^{*}\right)^{k-0}\left(-d\right)^{0}+P_{j}\binom{k}{1}\left(x_{j}{}^{*}\right)^{k-1}\left(-d\right)^{1}$$
$$+P_{j}\binom{k}{2}\left(x_{j}{}^{*}\right)^{k-2}\left(-d\right)^{2}+P_{j}\binom{k}{3}\left(x_{j}{}^{*}\right)^{k-3}\left(-d\right)^{3}$$
$$+P_{j}\binom{k}{4}\left(x_{j}{}^{*}\right)^{k-4}\left(-d\right)^{4}+P_{j}\binom{k}{5}\left(x_{j}{}^{*}\right)^{k-5}\left(-d\right)^{5}+\ldots+P_{j}\binom{k}{k-2}\left(x_{j}{}^{*}\right)^{2}\left(-d\right)^{k-2} \quad (99)$$
$$+P_{j}\binom{k}{k-1}\left(x_{j}{}^{*}\right)^{1}\left(-d\right)^{k-1}+P_{j}\binom{k}{k}\left(x_{j}{}^{*}\right)^{0}\left(-d\right)^{k}$$

$$E\left(X^{k}{}_{L}\right)=\sum_{j=1}^{m}\left\{\begin{array}{l}P_{j}\left(x_{j}{}^{*}\right)^{k}+P_{j}\binom{k}{1}\left(x_{j}{}^{*}\right)^{k-1}\left(-d\right)^{1}\\+P_{j}\binom{k}{2}\left(x_{j}{}^{*}\right)^{k-2}d^{2}+P_{j}\binom{k}{3}\left(x_{j}{}^{*}\right)^{k-3}\left(-d\right)^{3}\\+P_{j}\binom{k}{4}\left(x_{j}{}^{*}\right)^{k-4}d^{4}+P_{j}\binom{k}{5}\left(x_{j}{}^{*}\right)^{k-5}\left(-d\right)^{5}+\ldots+P_{j}\binom{k}{k-2}\left(x_{j}{}^{*}\right)^{2}\left(-d\right)^{k-2}\\+P_{j}\binom{k}{k-1}\left(x_{j}{}^{*}\right)^{1}\left(-d\right)^{k-1}+P_{j}\left(-d\right)^{k}\end{array}\right\} \quad (100)$$

$$E\left(X^{k}{}_{L}\right)=\left\{\begin{array}{l}\sum_{j=1}^{m}P_{j}\left(x_{j}{}^{*}\right)^{k}-\sum_{j=1}^{m}P_{j}\binom{k}{1}\left(x_{j}{}^{*}\right)^{k-1}d\\+\sum_{j=1}^{m}P_{j}\binom{k}{2}\left(x_{j}{}^{*}\right)^{k-2}d^{2}-\sum_{j=1}^{m}P_{j}\binom{k}{3}\left(x_{j}{}^{*}\right)^{k-3}d^{3}\\+\sum_{j=1}^{m}P_{j}\binom{k}{4}\left(x_{j}{}^{*}\right)^{k-4}d^{4}-\sum_{j=1}^{m}P_{j}\binom{k}{5}\left(x_{j}{}^{*}\right)^{k-5}d^{5}+\ldots+\sum_{j=1}^{m}P_{j}\binom{k}{k-2}\left(x_{j}{}^{*}\right)^{2}\left(-d\right)^{k-2}\\+\sum_{j=1}^{m}P_{j}\binom{k}{k-1}\left(x_{j}{}^{*}\right)^{1}\left(-d\right)^{k-1}+\left(-d\right)^{k}\sum_{j=1}^{m}P_{j}\end{array}\right\} \quad (101)$$

Then provided $\mu_{j}^{\bullet}<\infty$

$$E\left(X^{k}{}_{L}\right)=\left(\mu_{j}^{\bullet}\right)^{k}-\left(\mu_{j}^{\bullet}\right)^{k-1}\binom{k}{1}d+\left(\mu_{j}^{\bullet}\right)^{k-2}d^{2}\binom{k}{2}-\left(\mu_{j}^{\bullet}\right)^{k-3}d^{3}\binom{k}{3}+\left(\mu_{j}^{\bullet}\right)^{k-4}d^{4}\binom{k}{4}$$
$$-\left(\mu_{j}^{\bullet}\right)^{k-5}d^{5}\binom{k}{5}+\ldots+\left(\mu_{j}^{\bullet}\right)^{2}\binom{k}{k-2}\left(-d\right)^{k-2}+\left(\mu_{j}^{\bullet}\right)\binom{k}{k-2}\left(-d\right)^{k-1}+\left(-d\right)^{k} \quad (102)$$

Thus**,** the amount of claim size and the higher moments in the loss event is a function of the expected claim for the individual random risk and the deductible provided that $\mu_{j}^{\bullet}$ exists

7. Conclusion

Singularity models relating to general insurance business has been proposed. In particular, the work of Ogungbenle *et al.* (2020) evolved a distribution of claim size in a loss event in a flexible form. In this paper, we have successfully applied the dirac-delta singularity function in actuarial risk while focusing on characteristic function together with expected loss and higher moments of amount of claim size in a loss event which are governed by singularity potential. We arrived at the kth moment of the expected claim derivative from a convergent integral, provided with a core application to improve the traditional actuarial severity setting. The essence of correct estimation of frequency and severity of insurance claim is to allow an insurance firm to meet payments of claims as they occur and to fulfill solvency requirements. Reasonably, the impact of deductible is such that there will be lower number of claims advised when the deductible is enforced because a loss whose value is lower than the deductible would not produce any claim and in the event the scheme holder advises a claim, it is possible it would not result in payment going by the terms and conditions of the contract since that claim could be repudiated or the loss value could be determined to fall below the deductible. The policy holder would essentially pay a percentage of the first Naira premium for an excess or large deductible policy and takes responsibility for severity payments within the deductible domain of definition. Consequently, if the scheme holder thinks that it is a better risk than the average risk in underwriters' rating scale, then the policy holder could minimize the aggregate insurance costs. An underwriter could advise a scheme holder with severe loss experience to buy a deductible contract. This will shift a percentage of the severity-frequency exposure from the underwriter to the scheme holder. Since the scheme holder retains lower losses, the scheme holder has the incentive to optimally control claims in order to mitigate the number of losses. Further research work could be directed towards simulation of the expected claim.

References

Balcer Y., & Sahin I. (1979). Probabilistic models for pension benefits. *Journal of risk and Insurance, 46*, 99-123.

Dirac P.A.M (1930). The principle of quantum mechanics, oxford university press, London 58-67.

Frees E. W., Lee G., & Yang L. (2016). Multivariate frequency-severity regression models in insurance. *Risk*, *4*(1), 4.

Garrido J., Genest C., & Schulz J. (2016). Generalized linear models for dependent frequency and severity of insurance claims. *Insurance Mathematics and Economics*, *70*, 205-215.

Gomez-Deniz E. (2016). Bivariate credibility bonus-malus premiums distinguishing between two types of claims. *Insurance Mathematics and Economics, 70*, 117-124

Jack M., & Ormiston M.B. (1999). Analyzing demand for deductible insurance. *Journal of Risk and Uncertainty*, *18*, 223-230.

Khuri A.I. (2004). Application of dirac's delta function in statistics. *International Journal of mathematical Education in science and technology, 35*(2), 185-195.

Kanwal R.P. (1998). Function theory and technique (2nd edition), Birkhauser, Boston, pp. 1-17

Lui H., & Wang R. (2017). Collective risk models with dependence uncertainty. *ASTIN Bulletin*: *The Journal of the IAA*, *47*(2),361-389.

Mohammed H.R. (2011). Dirac's function and an its application in solving some problems in mathematics. *Journal of Al-Qadisiya for computer Science and Mathematics, 3*(3), 1-55.

Ogungbenle, M. G., Adeyele, J.S., & Mesioye A.E. (2020). Modeling moments of insurance claim size under dirac-delta Function. *Daffodil International University Journal of Science and Technology*, *15*(1), 43-51.

Onural L. (2006). Impulse function over curve and surface and their application to diffraction. *Journal of mathematics analysis and application*, *322*(1), 18-27.

Park S.C., Kim J.H., & Ahn J.Y. (2018). Does hunger for bonuses drive the dependence between claim frequency and severity?. *Insurance Mathematics and Economics*, *83*, 32-46.

Pazman A., & Pronzato L. (1996). A dirac-fraction method for densities of nonlinear statistic and for marginal densities in nonlinear regression, *Statistics & Probability letters*, *26*, 157.

Saichev A.I., & Woyczynski W.A. (1997). Distribution in the physical and engineering sciences, bodton MA, Brikhauser. pp. 1-38.

Sakthivel K.M., & Rajitha C.S. (2017). Artificial intelligence for estimation of future claim frequency in non-life insurance. *Global Journal of Pure and Applied mathematics, 13*(6),1701-1710.

Salansnich (2014). Quantum physics of light and matter unitext for physics springer publishing international Switzerland. Available at http//eds.cern.ch/1707567/ files /978-3-319-05179-6, bookbackmatter.pdf, pp.179-195.

Sastry S.S. (2009). Advanced engineering mathematics. PHI learning Ltd, New Delhi, pp 36-57.

Schlesinger H. (1985). Choosing a deductible for insurance contracts: Best or worst insurance policy?. *Journal of Risk and Insurance*, *52*(3), 522-527.

Schlesinger H. (1981). The optimal level of deductivity in insurance contracts. *Journal of Risk and insurance*. *48*(3), 465-481.

Thogersen J. (2016). Optimal premium as a function of the deductible. Customer analysis and portfolio characteristics. *Risks*, *4*, 1-19.

Tse Y.K. (2009). Non-life actuarial models, Cambridge University press, U.K, pp, 66-75.

Valecky J.I.R.I(2016). Modeling claim frequency in vehicle insurance. *Acta Universitatis agriculturae et Silviculturae Mendelianae Brunensis*, *64*(2), 683-689.

Venter, Mojor & Kreps (2006). Marginal decomposition of risk measures. *Astin Bulletin, 36*(2), 375-413.

Woodard J.D., & Yi J. (2018). Estimation of insurance deductible demand under endogenous premium rate. *Journal of risk and insurance*. *9999*, 1-24.

Zhang L. (2018). On the calculus of dirac-delta function with some application. *International Journal of mathematics trends and technology*, *56*(2), 256-270.