MODELLING OF SOLID WASTE MANAGEMENT : A DETERMINISTIC APPROACH

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Abstracts
The mathematical model of solid waste management is formulated following the pattern of the epidemiological model to control the menace of improper waste management and improve the revenue generated from Solid waste management. This study used nine systems of equations to represent the process of waste management from generation to collection, recycling, incineration and wealth creation. In this study the process of waste management is considered. Waste management entails the collection, treatment, recycling and eventual disposal. The New Generation matrix and Jacobian matrix were employed to calculate the reproduction number R_0*, the relationship between* R_0*, the waste control* ω *and rate of incinerating waste from residential area* b_2 *as well as the rate* b_1 *at which waste from residential sources are sent to landfill are well established. It was discovered that the reproduction number* R_0 *increases with increase in the quantity of waste incinerated and decreases with an increase in the control strategies* ω*. Regulation on the control and management of waste from residential and other sources will in no small measure control the menace of improper waste management.*

Keyword: Biodegradable, Incineration, Solid Waste, Decomposition.

Introduction
Waste can be defined as unwanted or unusable material, substances, or by-products (Swamy, 2022). Waste disposal is the process or system for getting rid of unwanted material by burying, burning, or dropping it in the sea. Waste management refers to the process of managing discarded waste materials that had served their purpose and are no longer useful. Waste management involves collecting solid waste materials, processing them and disposing them. According to (Senzige *et al.,* 2014), the main reason for studying waste management is to understand the intricacies involved in order to aid solid waste management planning. Wastes can be classified based on the following are generated from several sources such as domestic, industrial, agricultural, and commercial activities. Domestic Wastes are the waste materials produced from our households in our daily activities, it is usually referred to as refuse. About 90 percent of domestic waste is directly dumped on land thereby increasing land or soil pollution.

All Industries generate waste materials. The wastes typically include ashes, rubbish, building material wastes, toxic wastes, metal containers, plastic containers, paints, oils, and other complex synthetic materials. Modern techniques employed in agriculture and the use of a variety of chemicals have contributed to the production of large quantities of agricultural waste. A lot of waste are generated from commercial establishments such as restaurants, hotels, markets, offices, printing shops, auto repair shops, medical institutions and hospitals. There are mainly two types of wastes:- The waste which can be decomposed by the action of microorganisms are called biodegradable wastes, examples include domestic sewage, newspapers, and vegetable matters and other form of waste that can decay or decompose. The wastes which cannot be decomposed easily are called non-biodegradable wastes. They do not undergo decomposition,examples include polythene bags, plastics, glass, aluminum cans, iron nails, and DDT.

Senzige and Makinde (2016) presented a mathematical model of the effects of population dynamics on solid waste generation and treatment. The model was developed by grouping the population into three age classes and each class considered to have its own solid waste generation rate and natural death rate. The population is assumed to increase through birth and migration. Both the analytical and numerical results confirm that solid waste generation increases with increasing population growth. The sensitivity analysis reflects that increment in solid waste treatment efforts results in significant decrease in solid waste accumulation suggesting that with concerted treatment effort solid waste-free environment can be achieved.

Siddiqqi *et al.,* (2020) used a quantitative Waste to Energy Recovery Assessment (WERA) framework to analyse the feasibility of Waste to Energy in urban regions using emerging technologies such as gasification, pyrolysis, and refuse-derived fuel (RDF). Future policy measures of feed-in tariffs, payments for avoided pollution, and higher waste collection fees are used to evaluate if Waste to Energy systems can be made an autonomous investment. (O'connell, 2011), conducted a study on the factors that helped and hindered the involvement of the public in waste minimization and diversion. According to the report, common impediments to waste management include insufficient facilities, a value-action gap, a lack of faith in the government, costs associated with garbage diversion schemes, and unfavourable emotional reactions. Concern for the environment, societal norms for participation in activities, proper knowledge of the significance of trash diversion and minimization, the personal and collective benefits of waste minimization, and access to facilities for Â waste management Â were also mentioned as motivating factors.

The advantages of Pay As You Throw (P-A-Y-T) techniques in waste reduction and making sure that customers pay for garbage generated were emphasized by (Tornese, 2017). It was discovered that P-A-Y-T, a technology employed by the Internet of Things, enhanced data collection on garbage transmission and scheduling methods. (Momoh, 2019), formulated a deterministic mathematical model for waste management, to reduce and eliminate the consequences of poor waste management. It was concluded that waste reduction must be from the source and consumption of individuals must be reduced to their essential needs. (Senzige and Makinde, 2016), formulated an age-structured mathematical model, for waste generation, which divided the population into young, Adults, and elderly with different waste generation rates. It was concluded that solid waste accumulation increased with increased population and that increase in the natural decay rate of waste yield a significant decrease in solid waste accumulation.

Barnabas *et. al.,* (2019) reviewed 53 Solid Waste Management articles across the world focusing on characteristics, generation, collection, transportation, and other aspects of solid waste management. It was concluded after reviewing several works of literature that Minitab Software remains the best for statistical analysis of Solid Waste Management.

Materials and method

The total population $N(t)$ comprises of the municipal solid waste M(t) can be said to originate from two sources, the Residential sources $R(t)$ and the non-residential sources $Q(t)$. The wastes deposited at Dump-sites $D(t)$ can be sorted for Land-filling $L(t)$ and the incinerator $Z(t)$, The heat generated in the incinerator can be converted into Heat energy and Electricity $E(t)$, The ashes from the incinerator can be used in the Block production plant $B(t)$, The block production, the heat energy generated and the electricity generated can be used for

Wealth creation $W(t)$. The rate at which waste are generated at the residential and non-residential sources is β while ω serves as the control parameter for wastes generated from residential sources. Wastes from residential sources were sent to the landfill, incinerator and dumpsite at the rates b_1, b_2 *and* b_3 respectively. c_1, c_2 *and* c_3 represent the rates at which wastes were sent from non-residential sources to landfill, incinerator and dumpsite respectively. d_1 is the rate at which remnants from the incinerator was sent to the landfill d_2 is the rate at which heat from incinerator is converted to Electricity and d_3 is the rate at which ashes from incinerator is sent to block production machines. δ represent the decay of waste due to decomposition μ_1 *and* μ_2 represent reduction in the quantity of waste and non-waste materials respectively.

Model Assumptions.

i. A greater percentage of disposed waste are properly managed.
ii. The model considers only solid wastes generated from residential and non-residential sources.
iii. The reduction μ_1 in the quantity of waste differs from the reduction μ_2 the quantity of other variables that are not wastes.
iv. This model only considered energy generation from incineration.

(1)

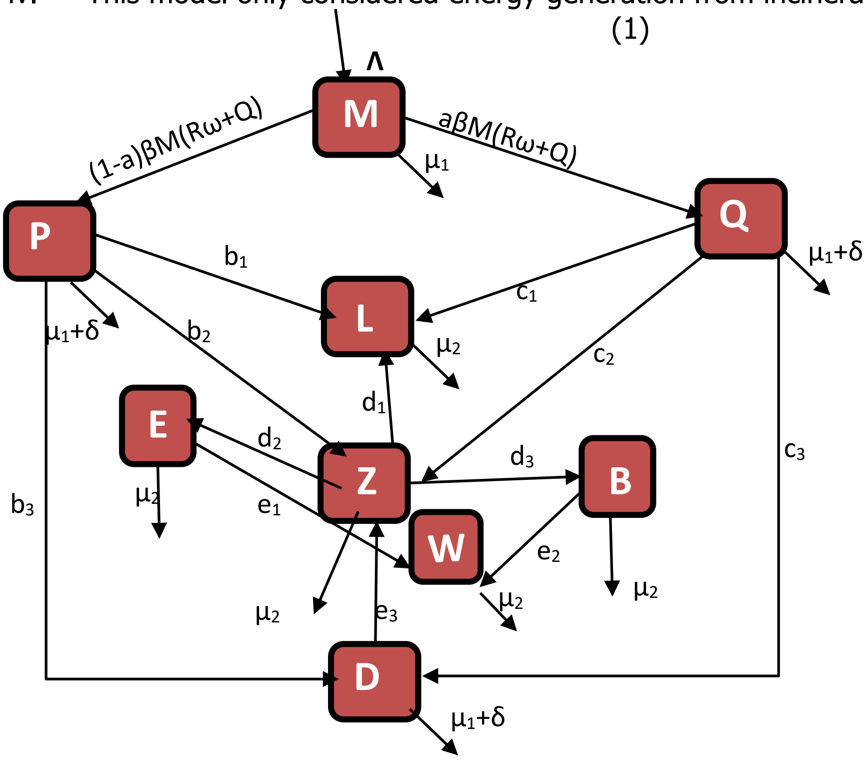

Figure 1. Schematic diagram

Table 1: Definition of Variable and Parameter

Var/Par	Description	Values	Source
P	Manufactured Goods for Residential purpose	1200	NBS
Q	Manufactured Goods for Non-Residential purposes	800	Estimated
M	Municipal solid waste	2000	NBS
D	Dump-Site	1256	NBS
L	Land-Fill site	800	Estimated
Z	Incinerator	1000	Estimated
E	Electricity generating plants	500	Estimated
B	Block producing plants	500	Estimated
W	Wealth creation	500	Estimated
Λ	Rate at which new products are used	2000	Ibikunle et al. 2022
β	Rate at which manufactured goods are used	0.008	Senzige & Makinde 2016
a	Proportion of solid waste from residential sources	0.5	Estimated
$1-a$	Proportion of solid waste from non-residential sources	0.5	Estimated
b_1	Rate at which wastes moves from Residential to landfill sites	0.04	Estimated
b_2	Rate at which wastes moves from Residential to Incinerator	0.04	Estimated
b_3	Rate at which wastes moves from Residential to Dump-sites	0.04	Estimated
c_1	Rate at which wastes moves from Non-residential to Landfill.	0.04	Estimated
c_2	Rate at which wastes moves from Non-residential to Incinerator	0.03	Estimated
c_3	Rate at which wastes moves from Non-residential to Dump-Site.	0.03	Estimated
d_1	Movement of Ashes fro Incinerator to landfill	0.06	Estimated
d_2	Part of burnt waste that converted to heat energy to generate Electricity	0.05	Estimated
d_3	Movement from incinerator to Block making machine	0.06	Estimated
e_3	Movement of waste from Dump-site to incinerator	0.05	Estimated.

μ_1	Reduction in the quantity of waste	0.04	Senzige&Makinde 2016
μ_2	Reduction in the quantity of other non-waste variables	0.03	Estimated
δ	Decay of waste due to decomposition	0.05	Senzige&Makinde 2016
e_1	Rate at which wealth is created from Energy generation	0.05	Estimated
e_2	Rate at which wealth is created from Block production	0.05	Estimated
ω	Rate at which wealth is created from Block production	0.04	Senzige&Makinde 2016

Existence and uniqueness of the solution

Derrick and Grossman theorem which is outlined in [1] shall be applied to verify the existence and uniqueness of solution of the model. Let D denotes the region $\| t-t_0 \| \le a, \| x-x_0 \| \le b, x=(x_1,x_2,...,x_n), \quad x_0=(x_1 0, x_2 0,...,x_n 0)$ and suppose $f(t,x)$ satisfies the Lipschitz condition $\| f(t,x_1)-f(t,x_2) \| \le k \| x_1-x_2$ The pairs (t,x_1) and (t,x_2) belong to D and k is a positive constant, hence there is a constant $\delta > 0$ such that there exists a unique continuous vector solution $x(t)$ of the system in the interval $t-t_0 \le 0.$ It is important to note that the condition is satisfied by the requirement that $\frac{\partial f_i}{\partial f_j}, i, j = 1,2,...,$ be continuous and bounded in D. We shall return to the model (1) and we consider the region $0 \le a \le \Re$. We look for a bounded solution in this region and whose partial derivatives satisfy $\delta \le a \le 0,$ where a and δ are constants.

Proof In order to proof the existence and uniqueness of the model , it is required to show that the each equation satisfies the Lipshictz condition, by expressing each compartment with respect to each of the state variables to satisfy the Lipsthitz condition.

Equations are renamed from $f_1 - f_8$

$$\left.\begin{aligned}
\frac{dM}{dt} &= \Lambda - \beta(\omega P + Q)M - \mu_1 M \\
\frac{dP}{dt} &= (1-a)\beta(\omega P + Q)M - k_1 P \\
\frac{dQ}{dt} &= a\beta(\omega P + Q)M - k_2 Q \\
\frac{dD}{dt} &= b_3 P + c_3 Q - k_3 D \\
\frac{dL}{dt} &= b_1 P + c_1 Q + d_1 Z - \mu_2 L \\
\frac{dZ}{dt} &= b_2 P + c_2 Q + e_3 D - k_4 Z \\
\frac{dE}{dt} &= d_2 Z - k_5 E \\
\frac{dB}{dt} &= d_3 Z - k_6 B \\
\frac{dW}{dt} &= e_1 E + e_2 B - \mu_2 W
\end{aligned}\right\} \quad (2)$$

Rewriting the first compartment of (2) as

$$\frac{dM}{dt} = \Lambda - \beta(\omega P + Q)M - \mu_1 M \quad (3)$$

$$\| f_1(M_1,t) - f_1(M_2,t) \| \le L_1 \| M_1 - M_2 \| \quad (4)$$

$$= \| f(\Lambda - \beta(\omega P + Q)M_1 - \mu_1 M_1) - f(\Lambda - \beta(\omega P + Q)M_2 - \mu_1 M_2) \| \quad (5)$$

gives

$$\left.\begin{aligned}
&\| (\Lambda - \beta(\omega P + Q)M_1 - \mu_1 M_1) - (\Lambda + \beta(\omega P + Q)M_2 + \mu_1 M_2) \| \\
&= \| (-1)(\beta(\omega P + Q)(M_1 - M_2) + \mu_1 (M_1 - M_2)) \| \\
&\le |-1| \| (\beta(\omega P + Q) + \mu_1)(M_1 - M_2) \| \\
&= (\beta(\omega P + Q) + \mu_1) \| (M_1 - M_2) \|
\end{aligned}\right\} \quad (6)$$

now, using the limiting values $P \le \frac{\Lambda}{\mu_1}$, $Q \le \frac{\Lambda}{\mu_1} \in \Omega$. It therefore, follows from (2) that

$$\| f_1(M_1,t) - f_1(M_2,t) \| \le \beta(\frac{\Lambda}{\mu_1} + \frac{\alpha\Lambda}{\mu_1} + \mu_1 \| M_1 - M_2 \| \quad (7)$$

the above equation implies that $f_1(M,t)$ is Lipschitz continuous with Lipschitz constant

$$L_1 = \left(\frac{\beta\Lambda}{\mu_1}(1+\alpha) + \mu_1 \right)$$

. . .

Rewriting the ninth compartment of (2)

$$\frac{dW}{dt} = e_1 E + e_2 B - \mu_2 W \quad (8)$$

$$\| f_9(e_1E + e_2B - \mu_2W_1, t) - f_9(e_1E + e_2B - \mu_2W_2, t) \| \quad (9)$$

$$\| e_1E + e_2B - \mu_2W_1 - e_1E + e_2B - \mu_2W_2 \| \quad (10)$$

$$\left.\begin{array}{l} \| -\mu_2W_1 + \mu_2W_2 \| = \| (-1)(\mu_2W_1 - \mu_2W_2) \| \\ \leq |-1| \mu_2 \| W_1 - W_2 \| = \mu_2 \| W_1 - W_2 \| \end{array}\right\} \quad (11)$$

the above equation implies that $f_9(t, W)$ in (2) is Lipschitz continuous with Lipschitz constant

$$L_9 = \mu_2$$

Positivity of Solution.

Theorem: Let $\Omega = (M, P, Q, D, L, Z, E, B, W) \in R_+^9 : M_0 > 0, P_0 > 0, Q_0 > 0, D_0 > 0, L_0 > 0, Z_0 > 0, E_0 > 0, B_0 > 0, W_0 > 0$

Then the solutions of M, P, Q, D, L, Z, E, B, W are positive for all $t \geq 0$

Proof: From the system of differential equation (2) The first equation is considered thus:

$$\frac{dM}{dt} = \Lambda - \beta(\omega P + Q)M - \mu_1 M$$

and the following inequality holds

$$\frac{dM}{dt} + (\,\beta(\omega P + \alpha Q) - \mu_1)M > 0 \quad (12)$$

Using the method of integrating factors equation (50) becomes

$$\frac{d\{M \;\; exp(\int_0^t \beta(\omega P(\phi) + Q(\phi))d\phi - \mu_1 t)\}}{dt} \quad (13)$$

Integrating both sides of equation (51)

$$M(t)\{\; exp(\int_0^t \beta(\omega P(\phi) + Q(\phi))d\phi - \mu_1 t)\} > M(0) \quad (14)$$

dividing both sides of (52) by the exponential gives

$$M(t) > M(0) \;\{\; exp(-\int_0^t \beta(\omega P(\phi) + Q(\phi))d\phi - \mu_1 t)\} > 0 \forall t > 0 \quad (15)$$

It follows that $M(t) > 0$

The second equation is considered thus:

$$\frac{dP}{dt} = (1 - a)\beta(\omega P + Q)M - k_1 P \quad (16)$$

and the following inequality holds

$$\frac{dP}{dt} + (\,\beta(\omega P + Q)M - k_1 P > 0 \quad (17)$$

Using the method of integrating factors equation (55) becomes

$$\frac{d\{P \;\; exp(\int_0^t \beta(\omega P(\phi) + Q(\phi))d\phi - k_1 t)\}}{dt} \quad (18)$$

Integrating both sides of equation (56)

$$P(t)\{\ exp(\int_0^t \beta(\omega P(\phi)+\alpha P(\phi))d\phi - k_1 t)\} > P(0) \quad (19)$$

dividing both sides of (57) by the exponential gives

$$P(t) > P(0)\ \{\ exp(-\int_0^t \beta(\omega P(\phi)+Q(\phi))d\phi - k_1 t)\} > 0\ \ \forall\ \ t > 0 \quad (20)$$

It follows that $P(t) > 0$

. . .

$$\frac{dW}{dt} = e_1 E + e_2 B - \mu_2 W \quad (21)$$

becomes

$$W(t) > W(0)\{\ exp(\int_0^t \beta(e_1 E(\phi)+e_2 B(\phi))d\phi - \mu_2 t)\} > 0\ \ \forall\ \ t > 0 \quad (22)$$

It follows that $W(t) > 0$

Invariant Reg

To obtain the invariant region in which the solution is bounded , we first consider the total quantity of waste (N) where $N = M + P + Q + D + L + Z + E + B + W$

It then follows that the rate of change of the total quantity of waste

$$\frac{dN}{dt} = \frac{dM}{dt} + \frac{dP}{dt} + \frac{dQ}{dt} + \frac{dD}{dt} + \frac{dL}{dt} + \frac{dZ}{dt} + \frac{dE}{dt} + \frac{dB}{dt} + \frac{dW}{dt} \quad (23)$$

$$\frac{dN}{dt} = \Lambda - \mu N - \delta(P + Q + D)$$

In the absence of wastage and loss during movement of waste from one point to another $\delta = 0$ equation(80)

$$\frac{dN}{dt} = \Lambda - \mu N \quad (24)$$

Then, the following differential inequality holds

$$\frac{dN}{dt} + \mu N \le \Lambda \quad (25)$$

$$\frac{dy}{dt} + P(y) = Q \quad (26)$$

Using the method of integrating factor

Then

$$\frac{d(N \exp \mu t)}{dt} \le \Lambda \exp \mu t \quad (27)$$

Integrating both sides of (84) gives

$$\int dN\ e^{\mu t} dt \le \int \Lambda e^{\mu t} dt \quad (28)$$

$$\int \frac{dN}{\mu N} \leq \int \Lambda e^{\mu t} dt \qquad (29)$$

$$N(t)e^{(\mu t)} \leq \frac{\Lambda}{\mu} e^{\mu t} + C \qquad (30)$$

Using the time at initial condition when time(t) is zero it can then be obtained that

$$N(0) \leq \frac{\Lambda}{\mu} + C; C = N(0) - \frac{\Lambda}{\mu}, \qquad (31)$$

substituting (88) into (87), the resulting equation gives

$$\left.\begin{aligned} N(t)e^{\mu t} &\leq \frac{\Lambda}{\mu} e^{\mu t} + N(0) - \frac{\Lambda}{\mu} \\ N(t) &\leq \frac{\Lambda}{\mu} + N(0)e^{-\mu t} - \frac{\Lambda}{\mu} e^{-\mu t} \\ N(t) &\leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}) \end{aligned}\right\} \qquad (32)$$

It therefore follows that

$$N(t) \leq \frac{\Lambda}{\mu} + N(0)e^{-\mu t} - \frac{\Lambda}{\mu} e^{-\mu t} \qquad (33)$$

If $N(0) \leq \frac{\Lambda}{\mu}$ then $N(t) \leq \frac{\Lambda}{\mu}$ therefore the region Ω is positively invariant. Further, if $N(0) \geq \frac{\Lambda}{\mu}$ then the solution either enters Ω in a finite time or $N(t) \to \frac{\Lambda}{\mu}$ asymptotically. Hence, the feasible region is attracting, which implies that all the solutions initiated in R_{+}^{11} eventually enter Ω. It suffices to consider the dynamics of solid waste management governed by the system of ordinary differential equation (2) in a feasible region Ω where the model is considered to be mathematically and sociologically well posed.

Existence of Waste-Free Equilibrium (WFE)

At the Waste-Free Equilibrium (WFE) point

$$\frac{dM}{dt} = \frac{dP}{dt} = \frac{dQ}{dt} = \frac{dD}{dt} = \frac{dL}{dt} = \frac{dZ}{dt} = \frac{dE}{dt} = \frac{dB}{dt} = \frac{dW}{dt} = 0$$

And Let

$$(M, P, Q, D, L, Z, E, B, W) = (M_0, P_0, Q_0, D_0, L_0, Z_0, E_0, B_0, W_0)$$

$$\left.\begin{aligned}
&\Lambda - \beta(\omega P_0 + Q_0)M_0 - \mu_1 M_0 &&= 0\\
&(1-a)\beta(\omega P_0 + Q_0)M_0 - k_1 P_0 &&= 0\\
&a\beta(\omega P_0 + Q_0)M_0 - k_2 Q_0 &&= 0\\
&b_3 P_0 + c_3 Q_0 - k_3 D_0 &&= 0\\
&b_1 P_0 + c_1 Q_0 + d_1 Z_0 - \mu_2 L_0 &&= 0\\
&b_2 P_0 + c_2 Q_0 + e_3 D_0 - k_4 Z_0 &&= 0\\
&d_2 Z_0 - k_5 E_0 &&= 0\\
&d_3 Z_0 - k_6 B_0 &&= 0\\
&e_1 E_0 + e_2 B_0 - \mu_2 W_0 &&= 0
\end{aligned}\right\} \tag{34}$$

Let

$$P_0 = 0; Q_0 = 0; \text{and } D_0 = 0; \tag{35}$$

then from the first equation in (91)

$$M_0 = \frac{\Lambda}{\mu_1} \tag{36}$$

substituting (92) and (93) into (91) gives

$$D_0 = 0; L_0 = 0; Z_0 = 0; E_0 = 0; B_0 = 0; W_0 = 0. \tag{37}$$

Therefore the Waste-Free Equilibrium (W.F.E)for the model equation (??) is given as

$$(M_0, P_0, Q_0, D_0, L_0, Z_0, E_0, B_0, W_0) = (\frac{\Lambda}{\mu_1}, 0, 0, 0, 0, 0, 0, 0, 0) \tag{38}$$

Waste reproductive number for model (R_0)

FV^{-1} is called the "Next Generation Matrix" (92) can be written as

$$\mathbf{f_i(x)} = \begin{pmatrix} (1-a)M_0\beta(P_0\omega + Q_0) \\ aM_0\beta(P_0\omega + Q_0) \\ 0 \end{pmatrix} \tag{39}$$

$$\mathbf{v_i(x)} = \begin{pmatrix} k_1 P_0 \\ k_2 Q_0 \\ k_3 D_0 - b_3 P_0 - c_3 Q_0 \end{pmatrix} \tag{40}$$

$$\mathbf{F(x)} = \begin{pmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial Q} & \frac{\partial f_1}{\partial D} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial Q} & \frac{\partial f_2}{\partial D} \\ \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial Q} & \frac{\partial f_3}{\partial D} \end{pmatrix} \tag{41}$$

$$\mathbf{V(x)} = \begin{pmatrix} \frac{\partial v_1}{\partial P} & \frac{\partial v_1}{\partial Q} & \frac{\partial v_1}{\partial D} \\ \frac{\partial v_2}{\partial P} & \frac{\partial v_2}{\partial Q} & \frac{\partial v_2}{\partial D} \\ \frac{\partial v_3}{\partial P} & \frac{\partial v_3}{\partial Q} & \frac{\partial v_3}{\partial D} \end{pmatrix} \tag{42}$$

$$\mathbf{F(x)} = \begin{pmatrix} aM_0\omega\beta & aM_0\beta & 0 \\ (1-a)M_0\omega\beta & (1-a)M_0\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{43}$$

$$\mathbf{V(x)} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ -b_3 & -c_3 & k_3 \end{pmatrix} \tag{44}$$

$$\mathbf{V^{-1}} = \begin{pmatrix} \frac{1}{k_1} & 0 & -\frac{b_3}{k_1k_3} \\ 0 & \frac{1}{k_2} & -\frac{c_3}{k_2k_3} \\ 0 & 0 & \frac{1}{k_3} \end{pmatrix} \tag{45}$$

Substituting

$$|\mathbf{FV^{-1}} - \boldsymbol{\lambda}| = \begin{vmatrix} \frac{(1-a)\Lambda\omega\beta}{\mu_1k_1} - \lambda & \frac{(1-a)\Lambda\beta}{\mu_1k_2} & 0 \\ \frac{a\Lambda\omega\beta}{\mu_1k_1} & \frac{a\Lambda\beta}{\mu_1k_2} - \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} \tag{46}$$

$$\left(\begin{array}{l} \left((\frac{(1-a)\Lambda\omega\beta}{\mu_1k_1} - \lambda)left(-\lambda(\frac{a\Lambda\beta}{\mu_1k_2} - \lambda) - 0 \right) - \\ \left(\frac{(1-a)\Lambda\beta}{\mu_1k_2}(-\frac{a\Lambda\lambda\omega\beta}{\mu_1k_1}) \right) = 0 \end{array} \right) \tag{48}$$

$$\left. \begin{array}{l} \frac{a(1-a)\lambda\omega\Lambda^2\beta^2}{\mu_1k_1k_2} + \frac{(1-a)\beta\omega\Lambda\lambda^2}{\mu_1k_1} + \\ \frac{a\beta\Lambda\lambda^2}{\mu_1k_2} - \lambda^3 + \frac{a(1-a)\lambda\omega\Lambda^2\beta^2}{\mu_1k_1k_2} = 0 \end{array} \right) \tag{49}$$

$$\lambda^2\left(\frac{(1-a)\beta\omega\Lambda}{\mu_1 k_1}+\frac{a\beta\Lambda}{\mu_1 k_2}-\lambda\right) \quad (50)$$

$$\lambda^2=0, or \lambda=\left(\frac{(1-a)\beta k_2\omega\Lambda+a\beta\Lambda k_1}{\mu_1 k_1 k_2}\right) \quad (51)$$

$$\lambda^2=0, or \lambda=\left(\frac{\beta\Lambda(1-a)k_2\omega+ak_1}{\mu_1 k_1 k_2}\right) \quad (52)$$

The R_0 which is the spectral radius of FV^{-1} denoted by $\rho(FV^{-1})$ is given as

$$R_0=\left(\frac{\beta\Lambda\left[k_2\ \omega\,(1-a)+ak_1\right]}{\mu_1 k_1 k_2}\right) \quad (53)$$

Local stability analysis of the Waste Free Equilibrium(WFE)

The local stability analysis of the waste free equilibrium point can be analysed using the Jacobian matrix of the system of equations(2). The stability is then determined based on Descartes rule of signs on the eigenvalues of the Jacobian matrix.

Theorem: The Waste Free Equilibrium Point is said to be locally asymptotically stable, provided that the associated basic reproduction number is less than unity and unstable otherwise $R_0>1$.

Proof:The Jacobian matrix J of equation (2) is given as

$$\mathbf{J}=\begin{pmatrix} \beta(\omega P+Q)-\mu_1 & -\beta\omega M & -\beta M_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (1-a)\beta(\omega P+Q) & (1-a)(\beta\omega M-k_1) & ((1-a)\beta M) & 0 & 0 & 0 & 0 & 0 & 0 \\ a\beta(\omega P+Q) & a\beta\omega M & a\beta\omega M-k_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_3 & c_3 & -k_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & e_3 & k_4 & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & d_1 & -\mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_2 & -k_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_3 & 0 & k_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & -\mu_2 \end{pmatrix}$$

(54)

$$
\mathbf{J(\varepsilon_0)} = \begin{pmatrix}
-\mu_1 & \frac{-\beta\omega\Lambda}{\mu_1} & \frac{-\beta\Lambda}{\mu_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1-a)(\frac{-\beta\omega\Lambda}{\mu_1} - k_1) & ((1-a)\frac{-\beta\Lambda}{\mu_1}) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a\frac{-\beta\omega\Lambda}{\mu_1} & a\frac{-\beta\omega\Lambda}{\mu_1} - k_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_3 & c_3 & -k_3 & 0 & 0 & 0 & 0 & 0 \\
0 & b_2 & c_2 & e_3 & k_4 & 0 & 0 & 0 & 0 \\
0 & b_1 & c_1 & 0 & d_1 & -\mu_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & -k_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_3 & 0 & k_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & -\mu_2
\end{pmatrix} \quad (55)
$$

Using the characteristic equation $|J|_{\varepsilon_0} - \lambda I| = 0$

$$
\begin{vmatrix}
-\mu_1-\lambda & \frac{-\beta\omega\Lambda}{\mu_1} & \frac{-\beta\Lambda}{\mu_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1-a)(\frac{-\beta\omega\Lambda}{\mu_1} - k_1)-\lambda & ((1-a)\frac{-\beta\Lambda}{\mu_1}) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a\frac{-\beta\omega\Lambda}{\mu_1} & a\frac{-\beta\omega\Lambda}{\mu_1} - k_2-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_3 & c_3 & -k_3-\lambda & 0 & 0 & 0 & 0 & 0 \\
0 & b_2 & c_2 & e_3 & k_4-\lambda & 0 & 0 & 0 & 0 \\
0 & b_1 & c_1 & 0 & d_1 & -\mu_2-\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_2 & -k_5-\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_3 & 0 & k_6-\lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & -\mu_2-\lambda
\end{vmatrix} = 0
$$

(55)

$$
(-\mu_1-\lambda)(-\mu_2-\lambda)\begin{pmatrix}
(1-a)(\frac{-\beta\omega\Lambda}{\mu_1} - k_1)-\lambda & ((1-a)\frac{-\beta\Lambda}{\mu_1}) & 0 & 0 & 0 & 0 \\
a\frac{-\beta\omega\Lambda}{\mu_1} & a\frac{-\beta\omega\Lambda}{\mu_1} - k_2-\lambda & 0 & 0 & 0 & 0 \\
b_3 & c_3 & -k_3-\lambda & 0 & 0 & 0 \\
b_2 & c_2 & e_3 & -k_4-\lambda & 0 & 0 \\
b_1 & c_1 & 0 & d_1 & -\mu_2-\lambda & 0 \\
0 & 0 & 0 & d_2 & -k_5-\lambda & 0 \\
0 & 0 & 0 & d_3 & 0 & -k_6-\lambda
\end{pmatrix} = 0
$$

(56)

Clearly, two of the eigenvalues were obtained to be negative from $(-\mu_1-\lambda)(-\mu_2-\lambda)=0$.

Reducing the $7X7$ matrix in (56) gives

$$(1-a)(\frac{-\beta\omega\Lambda}{\mu_1}-k_1)-\lambda\begin{pmatrix} a\frac{-\beta\omega\Lambda}{\mu_1}-k_2-\lambda & 0 & 0 & 0 & 0 \\ c_3 & -k_3-\lambda & 0 & 0 & 0 \\ c_2 & e_3 & k_4-\lambda & 0 & 0 \\ c_1 & 0 & d_1 & -\mu_2-\lambda & 0 \\ 0 & 0 & d_2 & -k_5-\lambda & 0 \\ 0 & 0 & d_3 & 0 & k_6-\lambda \end{pmatrix}$$

$$-((1-a)\frac{-\beta\Lambda}{\mu_1})\begin{pmatrix} a\frac{-\beta\omega\Lambda}{\mu_1} & 0 & 0 & 0 & 0 \\ b_3 & -k_3-\lambda & 0 & 0 & 0 \\ b_2 & e_3 & k_4-\lambda & 0 & 0 \\ b_1 & 0 & d_1 & -\mu_2-\lambda & 0 \\ 0 & 0 & d_2 & -k_5-\lambda & 0 \\ 0 & 0 & d_3 & 0 & k_6-\lambda \end{pmatrix} \tag{57}$$

$$\left((1-a)(\frac{-\beta\omega\Lambda}{\mu_1}-k_1)-\lambda(a\frac{-\beta\omega\Lambda}{\mu_1}-k_2-\lambda)\right)-\left(((1-a)\frac{-\beta\Lambda}{\mu_1})(a\frac{-\beta\omega\Lambda}{\mu_1})\right)\begin{pmatrix} -k_3-\lambda & 0 & 0 & 0 \\ b_2 & e_3 & k_4-\lambda & 0 \\ b_1 & 0 & d_1 & -\mu_2-\lambda \\ 0 & d_2 & -k_5-\lambda & 0 \\ 0 & d_3 & 0 & k_6-\lambda \end{pmatrix}$$

(58)

$$\left((1-a)(\frac{-\beta\omega\Lambda}{\mu_1}-k_1)-\lambda(a\frac{-\beta\omega\Lambda}{\mu_1}-k_2-\lambda)\right)-\left(((1-a)\frac{-\beta\Lambda}{\mu_1})(a\frac{-\beta\omega\Lambda}{\mu_1})\right)(k_3-\lambda)(k_4-\lambda)(k_5-\lambda)(k_6-\lambda)$$

(59)

$$\begin{aligned} &(1-a)a(\frac{-\beta^2\omega\Lambda^2}{\mu_1{}^2})-(1-a)(\frac{-\beta\omega\Lambda k_2}{\mu_1}) \\ &-(1-a)(\frac{\beta\omega\Lambda\lambda}{\mu_1})-a(\frac{\beta\Lambda}{\mu_1}k_1)+k_1k_2+k_1\lambda-a\frac{\beta\lambda\Lambda}{\mu_1}+\lambda^2 \\ &+k_2\lambda-(1-a)a(\frac{\beta^2\omega\Lambda^2}{\mu_1{}^2})-\lambda^2+(a\frac{\beta\omega\lambda\Lambda}{\mu_1})+ \\ &(1-a)(\frac{\beta\Lambda\lambda}{\mu_1})(k_3k_4-\lambda(k_3+k_4)+\lambda^2)(k_5k_6-\lambda(k_5+k_6)+\lambda^2) \end{aligned} \tag{60}$$

$$\begin{aligned} &+k_3k_4k_5k_6+\lambda k_5k_6(k_3+k_4)+\lambda k_3k_4(k_5+k_6)+k_3k_4\lambda^2 \\ &+k_5k_6\lambda^2+\lambda^2(k_3+k_4)(k_5+k_6)+\lambda^4+\lambda^3(k_5+k_6)+(k_3+k_4)\lambda^3 \\ &+\lambda(\frac{\beta\omega\Lambda}{\mu_1}+\frac{\beta\Lambda}{\mu_1}+k_1+k_2)-(\frac{1-a\beta\omega\Lambda k_2}{\mu_1}-\frac{a\beta k_1\Lambda}{\mu_1}+k_1k_2) \end{aligned} \tag{61}$$

$$\lambda^4+\lambda^3(k_3+k_4)(k_5+k_6)+\lambda^2(k_3k_4+k_5k_6+k_4k_5+k_4k_6+k_3k_6+k_3k_5)$$
$$+\lambda(k_3k_4k_5+k_3k_4k_6+k_3k_5k_6+k_4k_5k_6)+k_3k_4k_5k_6+\lambda(\frac{\beta\omega\Lambda}{\mu_1}+ \tag{62}$$
$$\frac{\beta\Lambda}{\mu_1}+k_1+k_2)-(\frac{1-a\beta\omega\Lambda k_2}{\mu_1}-\frac{a\beta k_1\Lambda}{\mu_1}+k_1k_2)$$

$$a_1(k_3+k_4)(k_5+k_6);a_2=k_3k_4+k_5k_6+k_4k_5+k_4k_6+k_3k_6+k_3k_5;a_3=k_3k_4k_5+k_3k_4k_6+k_3k_5k_6+k_4k_5k_6)a_4=k_3k_4k_5k_6$$
$$Y=\frac{1-a\beta\omega\Lambda k_2}{\mu_1}-\frac{a\beta k_1\Lambda}{\mu_1}+k_1k_2);X=\frac{\beta\omega\Lambda}{\mu_1}+\frac{\beta\Lambda}{\mu_1}+k_1+k_2$$
$$(X\lambda^5-Y\lambda^4+a_1X\lambda^4-Ya_1\lambda^3+$$
$$Xa_2\lambda^3+Ya_2\lambda^2+Xa_3\lambda^2+Ya_3\lambda+Xa_4\lambda-Ya_4)=0 \tag{63}$$

$$(X\lambda^5+(a_1X-Y)\lambda^4+(Xa_2-Ya_1)\lambda^3+\quad(Xa_3-Ya_2)\lambda^2+(Xa_4-Ya_3)\lambda-Ya_4)=0 \tag{64}$$

$$k_3k_4k_5k_6\left((\frac{1-a\beta\omega\Lambda k_2}{\mu_1}-\frac{a\beta k_1\Lambda}{\mu_1}+k_1k_2)\right)=0 \tag{65}$$

$$k_3k_4k_5k_6\left(1-R_0\right) \tag{66}$$

It can be seen that some of the eigenvalues of the linearised Jacobian matrix are negative, real and distinct. Hence, the Waste-Free Equilibrium is locally asymptotically stable whenever $R_0<1$ and unstable otherwise.

Existence of Waste-Persistent Equilibrium (WPE)

If the waste persistent state of the equilibrium for model equation II is represented as
$(M,P,Q,D,L,Z,E,B,W)=(M^*,P^*,Q^*,D^*,L^*,Z^*,E^*,B^*,W^*)$
At the W.P.E. it was considered that

$$\frac{dM^*}{dt}=\frac{dP^*}{dt}=\frac{dQ^*}{dt}=\frac{dD^*}{dt}=\frac{dL^*}{dt}=\frac{dZ^*}{dt}=\frac{dE^*}{dt}=\frac{dB^*}{dt}=\frac{dW^*}{dt}=0 \tag{67}$$

and Let

$$\beta(\omega P^*+Q^*)=\lambda^* \tag{68}$$

$$\left.\begin{array}{ll}\Lambda-\lambda^{*}M^{*}-\mu_{1}M^{*} & =0\\(1-a)\lambda^{*}M^{*}-k_{1}P^{*} & =0\\a\lambda^{*}M^{*}-k_{2}Q^{*} & =0\\b_{3}P^{*}+c_{3}Q^{*}-k_{3}D^{*} & =0\\b_{1}P^{*}+c_{1}Q^{*}+d_{1}Z^{*}-\mu_{2}L^{*} & =0\\b_{2}P^{*}+c_{2}Q^{*}+e_{3}D^{*}-k_{4}Z^{*} & =0\\d_{2}Z^{*}-k_{5}E^{*} & =0\\d_{3}Z^{*}-k_{6}B^{*} & =0\\e_{1}E^{*}+e_{2}B^{*}-\mu_{2}W^{*} & =0\end{array}\right\} \tag{69}$$

From the first equation in (69) it can be obtained that

$$\Lambda-\lambda^{*}+\mu_{1}M^{*}=0 \tag{70}$$

$$M^{*}=\frac{\Lambda}{\lambda^{*}+\mu_{1}} \tag{71}$$

Substituting (71) into second and third (69), the result is given as

$$\left.\begin{array}{l}P^{*}=\dfrac{\lambda^{*}M^{*}}{k_{1}}=\dfrac{(1-a)\Lambda\lambda^{*}}{k_{1}(\lambda^{*}+\mu_{1})}\\Q^{*}=\dfrac{\lambda^{*}M^{*}}{k_{2}}=\dfrac{a\Lambda\lambda^{*}}{k_{2}(\lambda^{*}+\mu_{1})}\end{array}\right\} \tag{72}$$

Substituting (72) into (68) the result is given as

(73)

The above method can also be used in (73) and substituting (74) the following relations can be obtained for the other state variables.

$$\left.\begin{aligned}
M^* &= \frac{\Lambda}{\mu_1(R_0-1+1)} = \frac{\Lambda}{\mu_1(R_0)}; P^* = \frac{(1-a)\Lambda\mu_1(R_0-1)}{k_1(\mu_1(R_0))}; \\
& Q^* = \frac{a\Lambda\mu_1(R_0-1)}{k_2(\mu_1(R_0))} \\
D^* &= \frac{\Lambda\mu_1(R_0-1)\left((1-a)b_3k_2+ac_3k_1\right)}{k_1k_2k_3(\mu_1(R_0))} \\
E^* &= \frac{d_2\mu_1(R_0-1)\,\Lambda\left((1-a)b_2k_2k_3+ac_2k_1k_3+(1-a)b_3k_2+ac_3k_1\right)}{k_1k_2k_3k_4k_5(\mu_1(R_0))} \\
Z^* &= \frac{\mu_1(R_0-1)\,\Lambda\left((1-a)b_2k_2k_3+ac_2k_1k_3+(1-a)b_3k_2+ac_3k_1\right)}{k_1k_2k_3k_4(\mu_1(R_0))} \\
B^* &= \frac{d_3\mu_1(R_0-1)\,\Lambda\left((1-a)b_2k_2k_3+ac_2k_1k_3+(1-a)b_3k_2+ac_3k_1\right)}{k_1k_2k_3k_4k_6(\mu_1(R_0))} \\
L^* &= \frac{(1-a)b_1\Lambda\mu_1(R_0-1)}{k_1\mu_2(\mu_1(R_0))} + \frac{ac_1\Lambda\mu_1(R_0-1)}{k_2\mu_2(\mu_1(R_0))} \\
+ & \quad \frac{d_1v\,\Lambda\left((1-a)b_2k_2k_3+ac_2k_1k_3+(1-a)b_3k_2+ac_3k_1\right)}{k_1k_2k_3k_4\mu_2(\mu_1(R_0))} \\
W^* &= \frac{\mu_1(R_0-1)\,\Lambda\left((1-a)b_2k_2k_3+ac_2k_1k_3+(1-a)b_3k_2+ac_3k_1\right)}{\mu_1k_1k_2k_3k_4(\mu_1(R_0))}\left[\frac{e_1d_2}{k_5}+\frac{e_2d_3}{k_6}\right]
\end{aligned}\right\} \quad (74)$$

from the above equation, it was observed that the Waste-Persistent Equilibrium is locally asymptotically stable provided the associated R_0 is less than unity that is $R_0 < 1$.

Results and Discussion

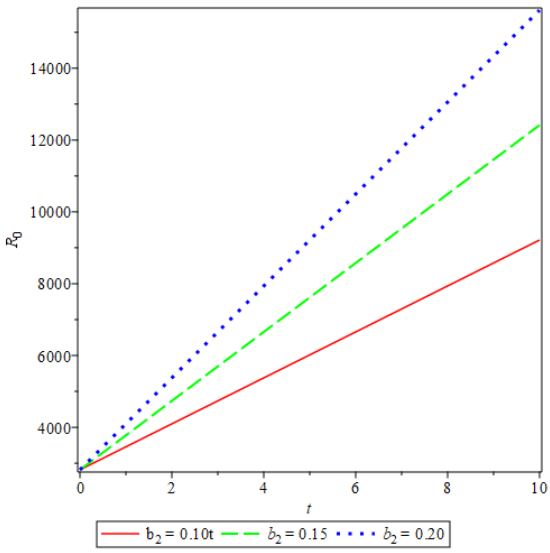

Figure 1: Change in Reproduction Number against time with varying rate of incineration

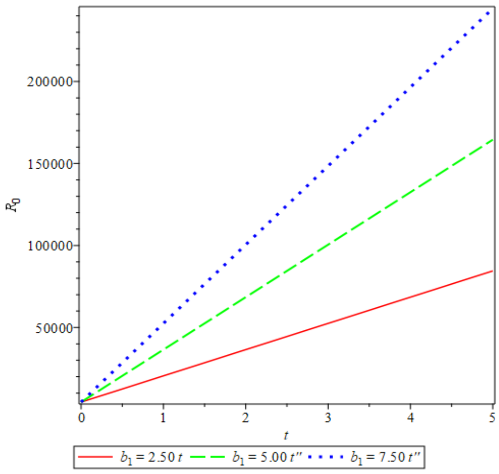

Figure 2: Change in Reproduction Number against time with varying rate of Land-filling

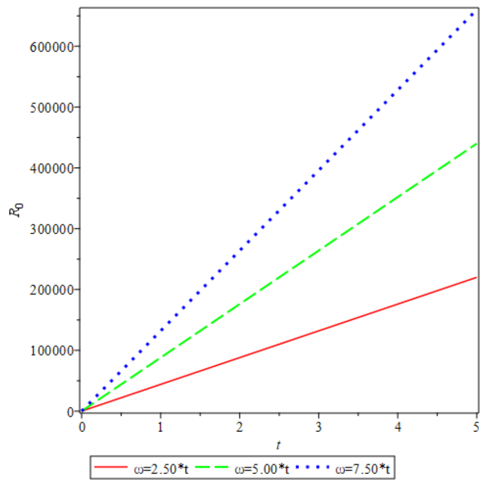

Figure 3: Change in Reproduction Number against time with varying control on waste generation.

Discussion

In this study, A mathematical model for managing solid waste, with a focus to creating wealth and generating energy, was formulated and analysed. The existence and uniqueness of the solutions to the model was established. To produce generate enough energy and wealth, it is vital to manage the rate of waste production in both households and industries. The basic reproduction number R_0 was calculated using the next generation matrix method. The sensitivity analysis was carried out on the model of solid waste management. equations When $R_0 < 1$, we demonstrated that the Waste-Free equilibrium is locally asymptotically stable. The relationship between the reproduction number and time with varying parameter was graphed using Maple software and the graph can be seen in the figures 2 to 4. It was observed from the graph that the rate b_1 and b_2 at which waste are been sent for incineration and land-filling increases the reproduction number increases. The control ω on waste generation will reduce the reproduction number as it increases

Conclusion

The model of solid waste management which is mathematical and sociologically well-posed was formulated and analysed. It was also established that the solution lies in the positive invariant region. The waste-free, waste-persistent equilibrium were obtained and the reproduction number was found using the next generation matrix. The model equations were further subjected to sensitivity analysis to know the most important parameter that will minimize wastage as well as maximize the resources generated from waste. The rate at which waste is sent to landfills and incineration depend largely on the control imposed on waste generation from residential and other sources. The implication of this is that waste generation can be properly managed if there are regulation on the control and management of waste from residential and other sources.

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