

## THE ALGEBRAIC AND TOPOLOGICAL CHARACTERIZATIONS OF THE CIRCLE GROUP

**OKONKWO CHUKWUKA CYPRAIN & SUNDAY ADESINA ADEBISI**

Department of Mathematics, University of Lagos, Akoka, Yaba, Lagos, Nigeria

**E-mail:** [adesinasunday@yahoo.com](mailto:adesinasunday@yahoo.com) **Phone No:** +2347041639013

### Abstract

*This study investigates the circle group—a key concept in contemporary mathematics through both algebraic and topological perspectives. Represented by  $T$  or  $S^1$ , the circle group encompasses all complex numbers with a modulus of one under multiplication, serving as an important link between the realms of abstract algebra and topology. From an algebraic standpoint, the circle group is classified as an abelian group, carefully related to notions such as homomorphisms, isomorphisms, cyclic subgroups, and the exponential map within Lie theory. Its structure reveals significant connections to the additive group, character theory and divisible groups. In terms of topology, it possesses an inherent manifold structure as a compact, connected, one-dimensional Lie group, positioning it as a fundamental element in the examination of continuous symmetries and topological groups. The interaction between its algebraic functions and topological characteristics illustrates the integration of discrete and continuous mathematics. Ultimately, the work highlights how the circle group showcases the key ideas across various mathematical disciplines, offering deep insights into both structural and functional aspects of mathematical theory.*

**Keywords:** Algebraic, Topological Characteristics, The Circle group

### Introduction

In the realm of mathematics, the concept of groups plays a crucial role in elucidating various types of symmetry and structural relationships across a multitude context (Milne, 2025). One particularly intriguing instance of this is the circle group, which emerges originally in the study of complex numbers, geometric transformations, and algebraic structures. This work aims to delve deeply into the properties of the circle group, exploring its algebraic structure. Furthermore, we will examine the significance of the circle group with various domains of mathematics, including its applications in fourier analysis, differential equations, particularly in the study of wave functions and oscillations. Through a comprehensive analysis, this report will highlight the ways in which the circle group not only serves as a fundamental building block for more complex mathematical concepts but also acts as a bridge interlinking diverse mathematical disciplines. By investigating its characteristics and implications, we aim to provide a clearer understanding of how groups, particularly the circle group, continue to shape mathematical idea and application across different fields.

The main objective of this work is to discuss in its entirety, the algebraic as well as the topological characteristic features for the circle group as an algebraic structure (Donaldson, 2025).

A typical element  $z$  of  $\mathbb{C}$  will be written  $z = x + iy$  where  $x, y \in \mathbb{R}$ . We identify  $z = x + iy$  with points  $(x, y)$  in the plane. Thus the absolute value  $|z|$  of  $z$  is defined by

$$|z| = \sqrt{x^2 + y^2}.$$

Note that since  $z\bar{z} = x^2 + y^2$  we also have:

$$|z| = \sqrt{z\bar{z}}.$$

We know from analytic geometry that  $|z|$  represents the distance from  $z$  to the origin  $0$  in the plane. The directed angle  $\theta$  that the segment from  $0$  to  $z$  makes with the positive side of the  $x$ -axis is called the *argument or polar angle* of  $z$ .

As in polar coordinates we write  $r = |z|$ . Then we have

$$\begin{aligned}x &= r \cos\theta, \\y &= r \sin\theta,\end{aligned}$$

and

$$z = r(\cos\theta + i\sin\theta)$$

Here,  $\theta$  represents the argument as given whenever the Argand diagram is well represented. We assume that our esteemed readers are well familiar with the exponential function  $x \mapsto e^x$  and where  $x \in \mathbb{R}$ . We extend the definition of this function from  $\mathbb{R}$  to  $\mathbb{C}$ .

**Definition 1.1:**

For  $z \in \mathbb{C}$  let  $z = x + yi$  where  $x, y \in \mathbb{R}$ , we define exponential function  $z \mapsto e^z$  by

$$e^z = e^{x+yi} = e^x(\cos y + i\sin y)$$

In particular, if  $\theta \in \mathbb{R}$  we have

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

From the above we have immediately the following:

**Theorem 1.1**

Every non-zero complex number  $z$  may be written uniquely in the form

$$z = re^{i\theta}$$

Where  $r = |z| > 0$  and  $0 \leq \theta < 2\pi$ .

Note that the expression  $e^{i\theta}$  is well-defined for all  $\theta \in \mathbb{R}$ .

**Theorem 1.2**

Let  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$  where  $r_i \geq 0$  and  $\theta_i$  are real numbers. Then  $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

**Theorem 1.3 (De Moivre's Theorem)** (Milne, 2025)

For all  $\theta \in \mathbb{R}$  and  $n \in \mathbb{R}$ , we have

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta),$$

equivalently,

$$(e^{i\theta})^n = e^{in\theta}.$$

**Definition 1.2:**

We define

$$T = \{z \in \mathbb{C} : |z|=1\};$$

$T$  is a group with respect to multiplication in  $\mathbb{C}$  and is called the **circle group**.

Furthermore, for every positive integer  $n$ ,  $|n| \leq n$ , where the leftmost  $n$  denotes the sum of  $n$  summands equal to the identity element. (Hestenes, *et al.*, 1984).

This classical absolute value and its square root are examples of absolute values, but not the square of the classical absolute value, which does not fulfill the triangular inequality.

An absolute value such that  $|x + y| \leq \max(|x|, |y|)$  is an ultrametric absolute value. An absolute value induces a metric (and thus a topology) by  $d(f, g) = |f - g|$ . (Dikranjan, *et al.*, 1990).

**The Geometric View In Algebra**

In mathematics, a geometric algebra (also known as Clifford algebra) is an algebra that can represent and manipulate geometric objects such as vectors (Joe. 1991). Geometric algebra is built out of two fundamental operations, addition and the geometric product. Multiplication of vectors results in higher-dimensional objects called multivectors.

The geometric product was first briefly mentioned by Hermann Grassmann, who was chiefly

interested in developing the closely related exterior algebra. In 1878, William Kingdon Clifford greatly expanded on Grassmann's work to form what are now usually called Clifford algebras in his honor (although Clifford himself chose to call them "geometric algebras"). (Hestenes, *et al.*, 1984).

Examples of geometric algebras applied in physics include the space time algebra (and the less common algebra of physical space). Geometric calculus, an extension of geometric algebra that incorporates differentiation and integration, can be used to formulate other theories such as complex analysis and differential geometry, e.g. by using the Clifford algebra instead of differential forms. Geometric algebra has been advocated, most notably by David Hestenes and Chris Doran, as the preferred mathematical framework for physics. Proponents claim that it provides compact and intuitive descriptions in many areas including classical and quantum mechanics, electromagnetic theory, and relativity. Geometric algebra has also found use as a computational tool in computer graphics and robotics (Usama, *et al.*, 2026; Saka *et al.*, 2025).

There are a number of different ways to define a geometric algebra. Hestenes's original approach was axiomatic, "full of geometric significance" and equivalent to the universal Clifford algebra. Given a finite-dimensional vector space  $V$  over a field  $F$  with a symmetric bilinear form (the inner product, e.g. the Euclidean metric)  $g : V \times V \rightarrow F$ , the geometric algebra of the quadratic space  $(V, g)$  is the Clifford algebra commonly defined as a quotient algebra of the tensor algebra though this definition is abstract, so the following definition is presented without requiring abstract algebra. In particular, its compact abelian structure and smooth manifold properties make it a cornerstone in the study of Lie groups (Adams, 1969) and duality principles.

### **Topological Structures**

The circle group is more than just an abstract algebraic object. It has a natural topology when regarded as a subspace of the complex plane. Since multiplication and inversion are continuous functions on  $C^\times$ , the circle group has the structure of a topological group. Moreover, since the unit circle is a closed subset of the complex plane, the circle group is a closed subgroup of  $C^\times$  (itself regarded as a topological group). One can say even more. The circle is a 1-dimensional real manifold, and multiplication and inversion are real-analytic maps on the circle. This gives the circle group the structure of a one-parameter group, an instance of a Lie group. In fact, up to isomorphism, it is the unique 1-dimensional compact, connected Lie group. Moreover, every  $n$ -dimensional compact, connected, abelian Lie group is isomorphic to  $T^n$  (Dikranjan, *et al.*, 1990)

### **Methodology**

#### **The Circle Group**

In mathematics, the circle group, denoted by  $T$  or  $S^1$ , is the multiplicative group of all complex numbers with absolute value 1, that is, the unit circle in the complex plane or simply the unit complex numbers.

$$T = \{z \in \mathbb{C} : |z| = 1\} \tag{1}$$

The circle group forms a subgroup of  $(\mathbb{C}, \times)$ , the multiplicative group of all nonzero complex numbers. Since  $(\mathbb{C}, \times)$  is abelian, it follows that  $T$  is as well. Then, we can express  $z$  as:

$$z = x + iy$$

Where  $x, y \in \mathbb{R}$ . And  $|z| = \sqrt{x^2 + y^2} = r$ , say we have that  $r^2 = x^2 + y^2$ .

By (1) above, it was stipulated and given that  $|z| = 1$ .

$$r^2 = x^2 + y^2 = 1.$$

Now, recall that if  $z = x + iy$ , then  $x, y \in \mathbb{R}$  and  $i = \sqrt{-1}$ , then in exponential (polar form)  $z = |z|e^{i\theta}$ , where  $\theta$  is the argument and is given by  $\theta = \tan^{-1} \frac{y}{x}$ .

Hence,  $z = |z|e^{i\theta} = |z|(\cos\theta + isin\theta)$ . But  $|z| = 1$ , by (1)

$$z = e^{i\theta} = \cos\theta + isin\theta$$

(2)

Furthermore, since  $z \in T$  and  $T$  as a group property,  $z^m \in T$  for some  $m \in \mathbb{R}$ . We have, by (2) that  $z^n = e^{-im\theta} = (\cos\theta + isin\theta)^m$ . By De Moivre's Theorem,

$$z^m = (\cos\theta + isin\theta)^m$$

Now, if  $z^n = 1$ , where  $m < n$ , then the  $n$ th root of unity would be given by:

$$z = 1^{1/n} = (\cos\theta + isin\theta)^{\frac{1}{n}} \Rightarrow z = \cos \frac{2k\pi}{n} + isin \frac{2k\pi}{n} \text{ for } k = 0, 1, 2, \dots, (n - 1).$$

Table 1

$k/n$	1	2	3	4	5
0	1	1	1	1	1
1	1	-1	$-1/2 + i\sqrt{3}/2$	1	0.31+i0.95
2	1	1	$-1/2 - i\sqrt{3}/2$	-1	-0.81+i0.59
3	1	-1	1	-1	-0.81-i0.59
4	1	1	$-1/2 + i\sqrt{3}/2$	1	0.31-i0.95

Table 2

$k/n$	1	2	3	4	5
5	1	-1	$-1/2 - i\sqrt{3}/2$	1	1
4	0.31-i0.95	1	$-1/2 + i\sqrt{3}/2$	1	1
3	-0.81-i0.59	-1	1	-1	1
2	-0.81+i0.59	-1	$-1/2 - i\sqrt{3}/2$	1	1

Table 3

$k/n$	1	2	3	4	5
1	0.31+i0.95	1	$-1/2 + i\sqrt{3}/2$	-1	1
5	1	-1	$-1/2 - i\sqrt{3}/2$	1	1
0	1	1	1	1	1
4	1	1	$-1/2 + i\sqrt{3}/2$	1	0.31-i0.95
3	1	-1	1	-1	-0.81-i0.59
2	1	1	$-1/2 - i\sqrt{3}/2$	-1	-0.81+i0.59
1	1	-1	$-1/2 + i\sqrt{3}/2$	1	0.31+i0.95
0	1	1	1	1	1

Table 4

$k/n$	<b>5</b>	<b>4</b>	<b>3</b>	<b>2</b>	<b>1</b>
0	1	1	1	1	1
1	0.31+i0.95	1	$-1/2+i\sqrt{3}/2$	-1	1
2	-0.81+i0.59	-1	$-1/2-i\sqrt{3}/2$	1	1
3	-0.81-i0.59	-1	1	-1	1
4	0.31-i0.95	1	$-1/2+i\sqrt{3}/2$	1	1
5	1	1	$-1/2-i\sqrt{3}/2$	-1	1

**Observations:**

1. The table labeled Table 1 to 4 contains  $n$  ranging from 1 to 5 and  $k$  ranging from 0 to 5
2. Both  $k$  and  $n$  starts from ascending order in Table 1
3. Both  $k$  and  $n$  starts from descending order in Table 2
4. In Table 3, the  $k$  is in descending order while  $n$  is ascending
5. In Table 4, the  $k$  is in ascending order while  $n$  is descending
6. We notice that when  $n=1$  all entries of  $k$  is 1
7. The values ranges from -1 to 1.
8. The roots of unity form a cyclic subgroup of T of order  $n$ .

**Proposition:** From the foregoing, the following are proposed :

(i.) For every value of  $n$  as well as  $k$ , the modulus is always equal to 1. Take for example, for  $(k, n) = (1, 3)$ , we are going to have that  $Z = -1/2+i\sqrt{3}/2$ , in which case,  $|Z| =$

$$|-1/2+i\sqrt{3}/2| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

(ii.) Whenever  $(n, k)$  is such that  $n = k$ ,  $Z$  is actually real and it is absolutely, a unit.

(iii) Both the real as well as the imaginary parts are bounded by  $[-1, 1]$

Define  $\mathbb{C}^* = \{\mathbb{C}/\{0\}\}$ , the multiplicative group of the complex numbers. The circle group is a subgroup of  $\mathbb{C}^*$ . Although, the circle group has infinite order, it certainly possesses many interesting finite subgroups. Now, suppose that  $H = \{-1, 1, i, -i\}$ . Then,  $H$  is a subgroup of the circle group. Here, note that the elements  $-1, 1, i, -i$  are exactly those complex numbers which satisfy the equation  $z^4 = 1$ , as a case example.

**The Exponential Map (Lie Theory)**

In the theory of Lie groups, the exponential map is a map from the Lie algebra  $\mathfrak{g}$  of a Lie group  $G$  to the group, which allows one to recapture the local group structure from the Lie algebra. The existence of the exponential map is one of the primary reasons that Lie algebras are useful tools for studying Lie groups. The ordinary exponential function of mathematical analysis is a special case of the exponential map when  $G$  is the multiplicative group of positive real numbers (whose Lie algebra is the additive group of all real numbers). The exponential map of a Lie group satisfies many properties analogous to those of the ordinary exponential function; however, it also differs in many important respects.

**Definitions:** Let  $G$  be the Lie group and  $\mathfrak{g}$  be its Lie algebra (thought of as the tangent space to the identity element of  $G$ ). The exponential map is a map.  $\exp: \mathfrak{g} \rightarrow G$

Which can be defined in several different ways.

**Definition:** The exponential of  $X \in \mathfrak{g}$  is given by  $\exp(X) = \gamma(1)$  where  $\gamma: X \rightarrow G$  is the unique group homomorphism of  $G$  whose tangent vector at the identity is equal to  $X$ .

It follows easily from the chain rule that  $\exp(tX) = \gamma(t)$ . The map  $\gamma$ , a group homomorphism

from  $(X, +)$  to  $G$ , may be constructed as the integral curve of either the right- or left-invariant vector field associated with  $X$ . That the integral curve exists for all real parameters followed by right- or left-translating the solution near zero (Kopperman., *et al* (1996)). We have a more concrete definition in the case of a matrix Lie group. The exponential map coincides with the matrix exponential and is given by the ordinary series expansion:

$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \dots$ , where  $I$  is the identity matrix. Thus, in the setting of matrix Lie groups, the exponential map is the restriction of the matrix exponential to the Lie algebra  $\mathfrak{g}$  of  $G$ .

### Applications

#### Signal Processing (Fourier analysis)

Let  $S^1$  stand for the set of all complex numbers  $z$  of absolute value one. This becomes a group under multiplication which is called the circle group. Any element  $z$  of this group can be written as  $z = e^{2\pi it}$  for a unique  $t \in [0, 1)$ . In view of this we can identify  $S^1$  with  $[0, 1)$  and there is a one to one correspondence between functions on  $S^1$  and functions on the real line  $\mathbb{R}$  that are 1-periodic, i.e. functions  $f$  satisfying  $f(t+1) = f(t)$  for all  $t \in \mathbb{R}$ . The most distinguished functions on  $S^1$  are the trigonometric functions  $e_k$  defined by:  $e_k(t) = (\cos 2\pi kt + i \sin 2\pi kt) = e^{2\pi i kt}$ .

Here  $k \in \mathbb{Z}$  the set of all integers. These functions are distinguished for several reasons.

First of all they are elementary, smooth functions which have been studied from ancient times. They are eigenfunctions of the one dimensional Laplacian:

$$\frac{d^2}{dt^2} e_k(t) = -4\pi^2 k^2 e_k(t).$$

For each  $k \in \mathbb{Z}$  the map

$$\chi_k : S^1 \rightarrow S^1, \chi_k(e^{2\pi i t}) = e_k(t)$$

is a homomorphism. Moreover, for each  $t$  the map

$$\varphi_t : X \rightarrow S^1, \varphi_t(k) = e_k(t)$$

is also a homomorphism. They form an orthonormal system in the sense that

$$(e_k, e_j) = \int_0^1 e_k(t) e_j(t) dt = \delta_{k,j}.$$

### The Euler Formula

Euler's formula, named after Leonard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number  $x$ , one has

$$e^{ix} = \cos x + i \sin x,$$

### Proofs

Using differentiation. This proof shows that the quotient of the trigonometric and exponential expression is the constant function one, so they must be equal (the exponential function is never zero, so this is permitted). Consider the function  $f(\theta) = \frac{\cos \theta + i \sin \theta}{e^{i\theta}} = e^{-i\theta} (\cos \theta + i \sin \theta)$  for real  $\theta$ . Differentiating gives by the product rule  $f'(\theta) = e^{-i\theta} (i \cos \theta - \sin \theta) - i e^{-i\theta} (\cos \theta + i \sin \theta) = 0$ . Thus,  $f(\theta)$  is a constant. Since  $f(0) = 1$ , then  $f(\theta) = 1$  for all real  $\theta$ , and thus  $e^{i\theta} = \cos \theta + i \sin \theta$ .

### Applications In Mathematics (Complex Number Theorem)

#### Interpretation of the formula

This formula can be interpreted as saying that the function  $e^{i\varphi}$  is a unit complex number, i.e., it traces out the unit circle in the complex plane as  $\varphi$  ranges through the real numbers. Here  $\varphi$  is the angle that a line connecting the origin with a point on the unit circle makes with the positive real axis, measured counterclockwise and in radians. A point in the complex plane can be represented by a complex number written in Cartesian coordinates. Euler's formula provides a means of conversion between Cartesian coordinates and polar coordinates. The polar form simplifies mathematics when used in multiplication or powers of complex numbers. Any

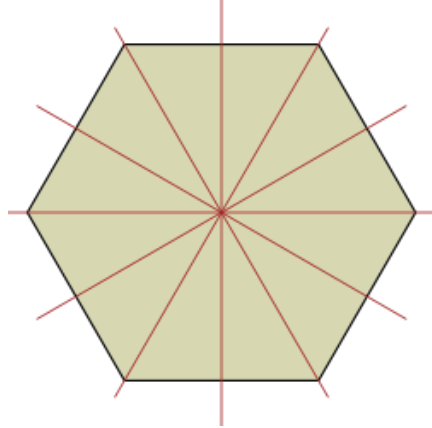
complex number  $z = x + iy$ , and its complex conjugate,  $\bar{z} = x - iy$ , can be written as

$$z = x + iy = |z|(\cos\phi + i\sin\phi) = re^{i\phi},$$

$\phi$  is the argument of  $z$ , i.e., the angle between the  $x$  axis and the vector  $z$  measured counterclockwise in radians, which is defined up to addition of  $2\pi$ . Many texts write  $\phi = \tan^{-1}y/x$  instead of  $\phi = \text{atan2}(y, x)$ , but the first equation needs adjustment when  $x \leq 0$ . This is because for any real  $x$  and  $y$ , not both zero, the angles of the vectors  $(x, y)$  and  $(-x, -y)$  differs by  $\pi$  radians, but have the identical value of  $\tan \phi = \frac{y}{x}$ . In topology, Euler's formula states that the imaginary exponential function  $t \mapsto e^{it}$  is a (surjective) morphism of topological groups from the real line  $\mathbb{R}$  to the unit circle  $S^1$ . Similarly, Euler's identity says that the kernel of this map is  $\tau\mathbb{Z}$ , where  $\tau = 2\pi$ . This superposition or linear combination is called the Fourier series.

### The Representation Theory

Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces, and studies modules over these abstract algebraic structures. In essence, a representation makes an abstract algebraic object more concrete by describing its elements by matrices and their algebraic operations (for example, matrix addition, matrix multiplication).



Representation theory studies how algebraic structures "act" on objects. A simple example is the way a polygon is transformed by its symmetries under reflections and rotations, which are all linear transformations about the center of the polygon (Joe, 1991)

### Results And Discussion

We have discussed the major features, properties as well as the characterizations of the circle group, based on the algebraic as well as the topological perspectives

### Conclusion

In this work, we have examined the circle group from both algebraic and topological viewpoints. Algebraically, we saw that the circle group is abelian, often represented as  $T = \{z \in \mathbb{C} : |z| = 1\}$ , or as the quotient group. Its structure reveals deep connections to cyclic groups, real numbers, and the exponential map. Topologically, we established that the circle group is a compact, connected, and Hausdorff space, with the structure of a one-dimensional Lie group. This makes it not only an important example in topology but also a fundamental object in the study of continuous symmetries. Through this characterization, we have seen how the circle group serves as a bridge between algebra and topology, and how it plays a key role in various areas of mathematics, such as harmonic analysis, Fourier series, and the theory of topological groups. Overall, the circle group stands as a simple yet powerful example of how abstract structures can unify different mathematical concepts, and its study provides a deeper appreciation for the interplay between algebraic operations and topological properties.

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