NUMERICAL TECHNIQUE FOR SOLVING FRACTIONAL GENERALIZED FISHER-KPP'S EQUATION USING COMPACT-FINITE-DIFFERENT METHODS TOGETHER WITH SHIFTED ULTRASPHERICAL COLLOCATION METHOD

ADESHOLA, A. D. [1] **, ISSA, K.**[2]**, BELLO, R. A.**[3]**, SULAIMAN, M. H.**[4]**, & ISHAQ, A. A.**[5]
[1,2,3,&4]Department of Mathematics and Statistics, Kwara State University,
Malete, Kwara State, Nigeria.
[5]Department of Physical Sciences, Al-Hikmah University, Ilorin, Nigeria.

Email: issa.kazeem@kwasu.edu.ng **Phone Number:** +234803 655 4437

Abstract
In this article, we propose an accurate numerical scheme for solving the fractional-order Fisher-Kolmogorov-Petrovsky-Piskunov (Fisher-KPP) equation, an important model in physics and engineering mathematics. The fractional derivative is defined in the sense of the Liouville-Caputo (LC) operator. The method combines a compact finite difference scheme (CFDS) for the temporal discretization with a spectral approach based on shifted ultraspherical polynomials for the spatial approximation. This approach transforms the original problem into a system of algebraic equations that can be solved efficiently. Theoretical analyses, including convergence and stability proofs, are provided to support the proposed scheme. Finally, numerical simulations are presented to demonstrate the accuracy and effectiveness of the method.

Keywords: Compact finite difference scheme (CFDS), Fisher's equation, Liouville-Caputo fractional derivative (LFD), shifted ultraspherical collocation method (SUCM), shifted ultraspherical polynomial.

Introduction
In recent years, mathematics has played a central role across all industries and major economic sectors. It is also essential in numerous real-life applications. As a natural consequence, it has become necessary to relate and connect real-world problems through mathematical modeling. In fields such as physics, chemistry, and biology, many reaction diffusion equations feature moving wavefronts, which are commonly encountered when modeling real-world phenomena (Abdeljawad & Baleanu, 2018, 2017). Reaction-diffusion models are powerful mathematical tools that describe how the concentration of one or more substances changes over space and time under the influence of two processes: first, local chemical reactions that convert substances into one another, and second, di_usion, which accounts for the spatial distribution of these substances. One of the most prominent examples is the classical Fisher-Kolmogorov-Petrovsky-Piskunov (Fisher-KPP) equation, which has found wide application in engineering and other scientific fields Newman (1980). However, the classical Fisher-KPP equation is not always sufficient for capturing complex behaviors in such models. The fractional-order Fisher-KPP equation o_ers a more accurate and effective framework for describing these phenomena Zhang & Liu (2014); Youssef *et al.* (2022). Fractional differential equations (FDEs), which involve fractional derivative operators, have proven to be more precise than classical models for capturing various real-world processes. Fractional calculus plays a significant role in modeling and understanding phenomena in scienti_c _elds Agarwal & El-Sayed (2020); Sweilam *et al.* (2020); Khader & Adel (2020); Mabrouk *et al.* (2024); Kedia *et al.* (2024); Nagy & Issa (2024). Despite their e_ectiveness, most fractional and variable-order differential equations lack exact analytical solutions. Therefore, numerical and approximate methods remain the primary tools for solving these equations. Various numerical techniques have been developed for this purpose, including the finite difference method Issa *et al.* (2022), the

wavelet collocation method Oruç (2020), the homotopy perturbation method Ağirseven & Oziş (2010), the residual power series method Al-Qurashi *et al.* (2017),, to mention but few.

Among the most powerful approaches for solving differential equations-whether partial, fractional, or of variable order-are spectral methods Atta *et al.* (2020). These methods are particularly valued for their ability to achieve highly accurate results with relatively few degrees of freedom Nagy & Issa (2024); Kumar *et al.* (2024); Sweilam *et al.* (2021). A key advantage of spectral methods lies in the orthogonality properties of special polynomials, such as shifted ultraspherical polynomials, which are used to approximate functions over a given interval [a; b] Issa *et al.* (2022); Izadkhah & Saberi-Nadja_ (2015). These polynomials play an important role in the spectral analysis of FDEs Nagy & Issa (2024). Therefore, the primary aim of this study is to investigate the fractional generalized Fisher-Kolmogorov-Petrovsky-Piskunov's equation using Liouville-Caputo (LC) derivative of the form:

$$\begin{cases} \dfrac{\partial \mathcal{Z}(x,t)}{\partial t} = \gamma \mathcal{D}_x^{\mu} \mathcal{U}(x,t) + \rho(1-\mathcal{U}(x,t)), & 1 < \mu \leq 2, 0 < x \leq 1, \\ \mathcal{U}(0,t) = \psi(t), & 0 < t \leq \mathcal{T} \\ \mathcal{U}(1,t) = 0, & 0 < t \leq \mathcal{T} \\ \mathcal{U}(x,0) = 0. \end{cases} \tag{1}$$

where $\mathcal{D}_x^{\mu}\mathcal{U}(x,t)$ is LC fractional derivatives operator, γ is the diffusive constant and the reactive constant is written as $0 \leq \rho \leq 1$. When $\mu = 2$, equation (1) becomes classical linear form of Fisher-KPP's equation. The exact solution to the classical form of equation (1) is given as follows:

$$\mathcal{U}(x,t) = 1 - \frac{\cosh\, x}{\cosh\, 1} - \frac{16}{\pi^2}\sum_{i=1}^{\infty} \frac{(-1)^i \cos\,(0.5\pi(2i-1)x}{(2i-1)(\pi^2(2i-1)^2+4)} \exp\left((-1-0.25\pi^2(2i-1)^2)t\right) \tag{2}$$

Several numerical approaches have been developed to solve Eq. (1), including the finite difference method Alshammari & Mashat (2017) and the Vieta-Lucas collocation method Youssef *et al.* (2022). In the present study, we aim to investigate the accuracy of the shifted ultraspherical collocation method, whose solutions generalize those obtained using Legendre polynomials when $\alpha = \frac{1}{2}$, Chebyshev polynomials of the second kind when $\alpha = 1$, and other classical orthoonal polynomials for different values of α. This method is combined with the compact finite difference technique for the discretization of the time derivative.

The structure of the paper is organized as follows: Section 2 introduces the properties of the LC fractional derivative and the shifted ultraspherical polynomials (SUP) utilized throughout the study. Section 3 describes the numerical scheme. Section 3.1 is devoted to the analysis of convergence and stability. In Section 4, we present numerical stimulation at various values of the parameters to demonstrate the effectiveness and accuracy of the scheme. Finally, Section 5 provides concluding remarks and summarizes the main findings.

2. Preliminaries

In what follows, we present essential definitions and mathematical preliminaries related to the fractional derivative (FD), along with key properties of ultraspherical polynomials, which are fundamental to the development of the methods proposed in this paper.

Definition 1 *(LC fractional derivative) Podlubny (1999)*
Suppose $\mu \in \mathbb{R}$ and $x \in [a,b]$, then the LC fractional-order derivative operator $\mathcal{D}_a^{\mu} f(x)$ of order μ is defined as follows:

$$\mathcal{D}_a^{\mu} f(x) = \frac{1}{\Gamma(n-\mu)} \int_0^x \frac{f^n(t)}{(x-t)^{\mu+1-n}} dt, n-1 < \mu \leq n, n \in \mathbb{N}, \tag{3}$$

Theorem 2.1 Let $\mathcal{U}(x)$ and $\mathcal{V}(x)$ be two functions defined on $[a, b]$ such that $\mathcal{D}_a^{\mu}\mathcal{U}(x)$ and $\mathcal{D}_a^{\mu}\mathcal{V}(x)$ exist almost everywhere. Also, let ν and $\lambda \in \mathbb{R}$. Then, $\mathcal{D}_a^{\mu}(\nu\mathcal{U}(x) + \lambda\mathcal{V}(x))$ exists almost everywhere, and

$$\mathcal{D}_a^{\mu}(\nu\mathcal{U}(x) + \lambda\mathcal{V}(x)) = \nu\mathcal{D}_a^{\mu}\mathcal{U}(x) + \lambda\mathcal{D}_a^{\nu}\mathcal{V}(x) \tag{4}$$

$\mathcal{D}^{\mu}G = 0$, G is a constant,

$$\mathcal{D}^{\mu}x^i = \begin{cases} 0, & \text{for } i \in \mathbb{N}_0 \text{ and } i < \lceil\mu\rceil \\ \frac{\Gamma(i+1)}{\Gamma(i+1-\mu)} x^{i-\mu}, & \text{for } i \in \mathbb{N}_0 \text{ and } i \geq \lceil\mu\rceil \end{cases}. \tag{5}$$

The function $\lceil\mu\rceil$ denotes smallest integer greater than or equal to μ, $\mathbb{N} = \{1,2 \dots\}$ and $\mathbb{N}_0 = \{0,1,2 \dots\}$.

The proofing of the theorem is contained in Podlubny (1999).

2.1 Shifted ultraspherical polynomials (SUP) $\mathbb{C}_i^{(\alpha)}(x)$

In this paper, the ultraspherical polynomials $\left\{\mathbb{C}_i^{(\alpha)}(x), i = 0,1, \dots\right\}$ are employed due to their generalization of several well-known families of orthogonal polynomials. Specifically, when $\alpha = \frac{1}{2}$, the ultraspherical polynomials reduce to the Legendre polynomials, and when $\alpha = 1$, it gives Chebyshev polynomials of the second kind. The ultraspherical polynomials $\left(\mathbb{C}_i^{(\alpha)}(\mathrm{x})\right)$ is an orthogonal polynomials of degree i in $x \in [-1,1]$ with respect to weight function $\omega(x) = (1-x^2)^{\left(\alpha-\frac{1}{2}\right)}$ is defined as follows Izadkhah & Saberi-Nadjafi(2015); Szegö (1975):

$$\mathcal{C}_i^{(\alpha)}(x) = \sum_{j=0}^{i} \frac{(-1)^j\, \Gamma(2\alpha + 2i - j)\, \Gamma\left(\alpha + \frac{1}{2}\right)}{(i-j)!\, \Gamma(j+1)\, \Gamma\left(i - j + \alpha + \frac{1}{2}\right) \Gamma(2\alpha)} x^{i-j}. \tag{6}$$

The recurrence form is given by

$$\mathcal{C}_i^{(\alpha)}(x) = \frac{1}{i}\left[2(i+\alpha-1)x\mathcal{C}_{i-1}^{(\alpha)}(x) - (i+2\alpha-2)\mathcal{C}_{i-2}^{(\alpha)}(x)\right], i \geq 2, \tag{7}$$

where $\mathbb{C}_0^{(\alpha)}(x) = 1, \mathbb{C}_1^{(\alpha)}(x) = 2\alpha x$.

The corresponding shifted form is defined as:

$$\mathcal{C}_i^{(\alpha)}(x) = \frac{1}{i}\left[2(i+\alpha-1)\left(\frac{2x-(a+b)}{b-a}\right)\mathcal{C}_{i-1}^{(\alpha)}(x) - (i+2\alpha-2)\mathcal{C}_{i-2}^{(\alpha)}(x)\right], i \geq 2, x \in [a, b] \tag{8}$$

where $\mathfrak{C}_0^{(\alpha)}(x) = 1, \mathfrak{C}_1^{(\alpha)}(x) = 2\alpha\left(\frac{2x-(a+b)}{b-a}\right)$. The explicit analytic form for the $\mathrm{SUP}^{\mathfrak{C}_i(\alpha)}(x)$ of degree i defined in the interval $[0,1]$ is given as:

$$\mathcal{C}_i^{(\alpha)}(x) = \sum_{j=0}^{i} \frac{(-1)^j\,\Gamma(2\alpha+2i-j)\,\Gamma\left(\alpha+\frac{1}{2}\right)}{(i-j)!\,\Gamma(j+1)\,\Gamma\left(i-j+\alpha+\frac{1}{2}\right)\Gamma(2\alpha)} x^{i-j}. \tag{9}$$

The orthogonality condition corresponding to the interval $x \in [a,b]$ is defined as follows:

$$\left\langle C_i^{(\alpha)}(x), C_j^{(\alpha)}(x)\right\rangle = \begin{cases} \int_{-1}^{1} (1-x^2)^{\left(\alpha-\frac{1}{2}\right)} \mathcal{C}_i^{(\alpha)}(x)\mathcal{C}_j^{(\alpha)}(x)dx = \begin{cases} 0, & \text{for } i \neq j \\ \dfrac{\pi 2^{1-2\alpha}\,\Gamma(i+2\alpha)}{i!\,\Gamma(\alpha) I^2 (i+\alpha)}, & \text{for } i = j \end{cases} \\ \int_{0}^{1} (x-x^2)^{\left(\alpha-\frac{1}{2}\right)} \mathcal{C}_i^{(\alpha)}(x)\mathcal{C}_j^{(\alpha)}(x)dx = \begin{cases} 0, & \text{for } i \neq j \\ \dfrac{\pi 2^{1-4\alpha}\,\Gamma(i+2\alpha)}{i!\,\Gamma(\alpha)\|^2 (i+\alpha)}, & \text{for } i = j, \end{cases} \end{cases} \tag{10}$$

Let $\mathcal{U}(x)$ be a square integrable function in the interval $[a,b]$, then

$$\mathcal{U}(x) = \sum_{i=0}^{\infty} \beta_i C_i^{(\alpha)}(x) \tag{11}$$

where the coefficients $\beta_n, n = 0,1,\dots,\mathcal{N}$ is defined as:

$$\beta_i = \begin{cases} \dfrac{i!\,[\Gamma(\alpha)]^2 (i+\alpha)}{\pi 2^{1-2\alpha}\,\Gamma(i+2\alpha)} \int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} \mathcal{U}(x)\mathcal{C}_i^{(\alpha)}(x)dx, & x \in [-1,1] \\ \dfrac{i!\,[\Gamma(\alpha)]^2 (i+\alpha)}{\pi 2^{1-4\alpha}\,\Gamma(i+2\alpha)} \int_{0}^{1} (x-x^2)^{\alpha-\frac{1}{2}} \mathcal{U}(x)\mathcal{C}_i^{(\alpha)}(x)dx, & x \in [0,1] \end{cases} \tag{12}$$

depending on the interval of equation (1).
Only the first $(\mathcal{N}+1)$-terms of shifted ultraspherical polynomials are needed in the approximation. Therefore, equation (11) becomes

$$\mathcal{U}(x) \approx \mathcal{U}_{\mathcal{N}}(x) = \sum_{n=0}^{\mathcal{N}} \beta_n C_n^{(\alpha)}(x). \tag{13}$$

Theorem 2.2 Let $\mathcal{U}_{\mathcal{N}}(x)$ be an approximate solution of equation (1), then LC fractional derivative of $\mathcal{U}_{\mathcal{N}}(x)$ is given as:

$$\mathcal{D}_a^{\mu}\left(\mathcal{U}_{\mathcal{N}}(x)\right) = \sum_{n=[\mu]}^{\mathcal{N}} \sum_{k=0}^{n-[\mu]} \beta_n \mathscr{H}_{n,k} x^{n-k-\mu}. \tag{14}$$

where

$$\mathscr{H}_{n,k} = \frac{(-1)^k\,\Gamma(2\alpha+2n-k)\,\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(k+1)\,\Gamma\left(n-k+\alpha+\frac{1}{2}\right)\Gamma(2\alpha)\,\Gamma(n+1-k-\mu)}. \tag{15}$$

See Issa *et al.* (2022) for the proof.

Numerical Scheme

In this section, we employ a combination of the compact finite difference scheme (CFDS) and the shifted ultraspherical collocation method to obtain a numerical solution of equation (1). We begin by deriving a time-discrete scheme based on CFDS. To this end, we define the time mesh points as $t_j = j\delta t$ for $j = 0,1,\dots,\mathcal{M}$, where $\delta t = \frac{\tau}{\mathcal{M}}$. For the spatial discretization, we

employ collocation points $\mathfrak{C}_{N-[\gamma]+1}^{(a)}(x_i)$, with $i = 0,1,\dots,\mathcal{N}-[\gamma]+1$.
Let

$$\mathcal{U}_{\mathcal{N}}(x,t) = \sum_{n=0}^{\mathcal{N}} \beta_n(t)\mathscr{C}_n^{(\alpha)}(x), \tag{16}$$

be the approximate solution of equation (1), therefore equation (1) becomes:

$$\frac{\partial \mathcal{U}_{\mathcal{N}}(x,t)}{\partial t} = \gamma \mathcal{D}_{0^+}^{u}\mathcal{U}_{\mathcal{N}}(x,t) + \rho(1-\mathcal{U}_{\mathcal{N}}(x,t)), \tag{17}$$

Now, applying Taylor's expansion, equation (17), becomes:

$$\delta\tau\mathcal{U}_{\mathcal{N}}(x_i,t_j) + \frac{\delta\tau}{2}\frac{\partial^2\mathcal{U}_{\mathcal{N}}(x_i,t_j)}{\partial t^2} + \mathcal{O}(\delta\tau^2) = \gamma\mathcal{Q}_{0+}^{\mu}\mathcal{U}_{\mathcal{N}}(x_i,t_j) + \rho\left(1-\mathcal{U}_{\mathcal{N}}(x_i,t_j)\right). \tag{18}$$

Differentiating equation (1) with respect to τ and substitute the resulting equation in equation (18), we obtain:

$$\delta\tau\mathcal{U}_{N_i}^{j} = \gamma\mathcal{D}_{0+}^{u}\mathcal{U}_{N_i}^{j} + \rho\left(1-\mathcal{U}_{N_i}^{j}\right) - \frac{\delta\tau}{2}\left[\gamma\left(\frac{\mathcal{D}_{0+}^{u}\mathcal{U}_{N_i}^{j} - \mathcal{D}_{0+}^{u}\mathcal{U}_{N_i}^{j-1}}{\delta\tau}\right) - \rho\left(\frac{\mathcal{U}_{N_i}^{j} - \mathcal{U}_{N_i}^{j-1}}{\delta\tau}\right)\right], \tag{19}$$

simplifying equation (19), we obtain:

$$A_1U_{N_i}^{j} - A_2\mathcal{D}_{0+}^{u}U_{N_i}^{j} = A_3U_{N_i}^{j-1} + A_2\mathcal{D}_{0+}^{u}U_{N_i}^{j-1} + \rho\delta\tau, \tag{20}$$

where $A_1 = \frac{2+p\sqrt{\tau}}{2}, A_2 = \frac{\gamma\delta\tau}{2}$ and $A_3 = \frac{2-p\sigma_1\tau}{2}$.
Substitute equations (13) and (14) in equation (20), we obtain

$$\begin{aligned} &A_1\sum_{n=0}^{\mathcal{N}} \beta_n^m\mathscr{C}_n^{(\alpha)}(x) - A_2\sum_{n-[\mu]}^{\mathcal{N}}\sum_{k=0}^{n-[\mu]} \beta_n^m\mathscr{K}_{n,k}x^{n-k-\mu} \\ &= A_3\sum_{n=0}^{\mathcal{N}} \beta_n^{m-1}\mathscr{C}_n^{(\alpha)}(x) + A_2\sum_{n=[\mu]}^{\mathcal{N}}\sum_{k=0}^{n-[\mu]} \beta_n^{m-1}\mathscr{K}_{n,k}x^{n-k-\mu} + \rho\delta\tau. \end{aligned} \tag{21}$$

At degree $\mathcal{N} = 3$, collocating equation (21) at $x = x_i, i = 0,1,\dots,\mathcal{N}-[\mu]+1$, and simplify the resulting equation, we obtain:

$$A_1\beta_0^m + B_1\beta_1^m + B_2\beta_2^m + B_3\beta_3^m = A_3\beta_0^{m-1} + B_4\beta_1^{m-1} - B_2\beta_2^{m-1} + B_5\beta_3^{m-1} + \rho\delta\tau, \tag{22}$$

where

$$\begin{aligned} B_1 &= A_1\mathscr{C}_1^{(\alpha)}(x_i) \\ B_2 &= -A_2x_i^{2-\mu}\mathscr{K}_{2,0} \\ B_3 &= A_1\mathscr{C}_3^{(\alpha)}(x_i) - A_2\left(x_i^{3-\mu}\mathscr{K}_{3,0} + x_i^{2-\mu}\mathscr{K}_{3,1}\right) \\ B_4 &= A_3\mathscr{C}_1^{(\alpha)}(x_i) \\ B_5 &= A_3\mathscr{C}_3^{(\alpha)}(x_i) + A_2\left(x_i^{3-\mu}\mathscr{K}_{3,0} + x_i^{2-\mu}\mathscr{K}_{3,1}\right) \end{aligned} \tag{23}$$

Substitute equation (16) into boundary conditions of equation (1), we obtain

$$\mathcal{U}_{\mathcal{N}}(0,t) = \sum_{n=0}^{N} \frac{(-1)^n\,\Gamma(n+2\alpha)}{n!\,\Gamma(2\alpha)} \beta_n(t) = \psi(t)$$
$$\mathcal{U}_{\mathcal{N}}(1,t) = \sum_{n=0}^{N} \frac{\Gamma(n+2\alpha)}{n!\,\Gamma(2\alpha)} \beta_n(t) = 0 \qquad (24)$$

This results in $\mathcal{N}+1$ linear algebraic equations, which are solved to determine the unknowns coefficients $\beta_{\mathrm{n}}^{m}, n = 1,2,\dots,\mathcal{N}$. Note, the initial values of $\beta_{\mathrm{n}}^{0}, n = 1,2,\dots,\mathcal{N}$ are obtained from equation (12).

3.1 Convergence and Stability Analysis

To examine the stability and convergence of the proposed technique, let Θ be an open and bounded domain in $\mathbb{R}^2$ and $\mathbb{L}_2^{\gamma,\delta}(\Theta)$ be a Hilbert space with the inner product

$$\langle \mathscr{F}(x), \mathcal{G}(x) \rangle = \int_{\Theta} \mathscr{F}(x)\mathcal{G}(x)dx \qquad (25)$$

Euclidean norm $\|\mathcal{F}(x)\| = \langle \mathcal{F}(x), \mathcal{F}(x) \rangle^{\frac{1}{2}}$ and Sobolev space as

$$H^{\mu}(\Theta) = \{\mathscr{F} \in \mathbb{L}_2^{\mu}(\Theta), \mathcal{D}^{\mu} \in L_2^{\mu}(\Theta)\} \qquad (26)$$

In the following, we present key lemmas that are crucial for analyzing the stability and convergence of the proposed scheme.

Lemma 3.1 *For any* $\mathcal{F}, \mathcal{G} \in H^{\frac{\mu}{2}}(\theta)$, *then*

$$\langle \mathcal{D}_{a^+}^{u}\mathcal{F}, \mathcal{G} \rangle = \left\langle \mathcal{D}_{a^+}^{\frac{\mu}{2}}\mathcal{F}, \mathcal{D}_{b^-}^{\frac{\mu}{2}}\mathcal{G} \right\rangle$$
$$\langle \mathcal{D}_{b^-}^{u}\mathcal{F}, \mathcal{G} \rangle = \left\langle \mathcal{D}_{b^-}^{\frac{\mu}{2}}\mathcal{F}, \mathcal{D}_{a^+}^{\frac{\mu}{2}}\mathcal{G} \right\rangle, \ for\ 1 < \gamma < 2$$
$$\langle \mathcal{D}_{a^+}^{u}\mathcal{F}, \mathcal{D}_{b^-}^{u}\mathcal{F} \rangle = \cos(\mu\pi)\left\|\mathcal{D}_{a^+}^{u}\mathcal{F}\right\|^2 = \cos(\mu\pi)\|\mathcal{D}_{b^-}^{u}\mathcal{F}\|^2$$

Proof: The proof is provided in Ervin & Roop (2007).

Lemma 3.2 Given the functions $\mathcal{G}(x)$ and $\mathcal{D}_{a^+}^{\mu} \in H^{\mu}(\theta) \exists \delta\tau$ sufficiently small such that

$$\|\mathcal{G}(x)\| \le \left\|\mathcal{G}(x) + \mathcal{Q}_{a^+}^{\mu}\mathcal{G}(x) + \frac{\rho\delta\tau}{A_3}\right\|, \qquad (27)$$

where $\mathcal{Q} = \frac{A_2}{A_3} = \frac{\rho\delta\tau}{2-\rho\delta\tau}$

Proof: Using lemma 3.1, we obtain

$$
\begin{aligned}
&\|\mathcal{G}(x) + \mathcal{Q}\mathcal{Q}_{a^+}^{\mu}\mathcal{G}(x) + \frac{\rho\delta\tau}{A_3}\|^2 \\
&= \left\langle \mathcal{G}(x) + \mathcal{Q}\mathcal{Q}_{a^+}^{\mu}\mathcal{G}(x) + \frac{\rho\delta\tau}{A_3}, \mathcal{G}(x) + \mathcal{Q}\mathcal{Q}_{a^+}^{\mu}\mathcal{G}(x) + \frac{\rho\delta\tau}{A_3} \right\rangle \\
&= \|\mathcal{G}(x)\|^2 + 2\mathcal{Q}\left(\mathcal{D}_{a^+}^{\frac{\pi}{2}}\mathcal{G}(x), \mathcal{D}_{b^-}^{\mu}\mathcal{G}(x) \right\rangle + 2\frac{\rho\delta\tau}{A_3}\mathcal{G}(x) + \mathcal{Q}^2\left\| \mathcal{D}_{a^+}^{\frac{\mu}{2}}\mathcal{G}(x) \right\|^2 \\
&\quad + 2\frac{\rho\delta\tau}{A_3}\mathcal{Q}\|\mathcal{G}(x)\|^2 + \left(\frac{\rho\delta\tau}{A_3}\right)^2 \\
&= \|\mathcal{G}(x)\|^2 + 2\mathcal{Q}\cos\left(\frac{\mu}{2}\pi\right)\left\| \mathcal{D}_{a^+}^{\frac{\mu}{2}}\mathcal{G}(x) \right\|^2 + 2\frac{\rho\delta\tau}{A_3}\mathcal{G}(x) + \mathcal{Q}^2\left\| \mathcal{D}_{a^+}^{\frac{\mu}{2}}\mathcal{G}(x) \right\|^2 \\
&\quad + 2\frac{\rho\delta\tau}{A_3}\mathcal{Q}\|\mathcal{G}(x)\|^2 + \left(\frac{\rho\delta\tau}{A_3}\right)^2,
\end{aligned} \tag{28}
$$

for sufficiently small $\delta\tau$, the following terms are negligible $\mathcal{Q}^2, \delta\mathcal{Q}, \delta\tau^2$ and

$$\frac{2\rho\delta\tau}{A_3} + 2\mathcal{G}\cos\left(\frac{\mu}{2}\pi\right)\left\| \mathcal{D}_{a^+}^{\frac{\mu}{2}}\mathcal{G}(x) \right\|^2 \geq 0,$$

therefore

$$\|\mathcal{G}(x)\| \leq \left\| \mathcal{G}(x) + \mathcal{Q}\mathcal{D}_{a^+}^{\mu}\mathcal{G}(x) + \frac{\rho\delta\tau}{A_3} \right\|,$$

hence the proof.

Theorem 3.3 (See Youssef et al. (2022); Nagy & Issa (2024)). The scheme provided in equation (20) for Fisher-KPP equation (1) is unconditionally stable.
The proof is provided in Youssef *et al.* (2022).

Numerical Experiments

In this section, we apply the aforementioned technique to process and solve the proposed model using various values of the fractional order μ and the approximation order $\mathcal{N}$. The approximate solution is also compared with the exact solution when $\mu = 2$. Additionally, we set diffusion constant $\gamma = 1$ and the reactive constant $\rho = 1$ to observe their effects. The effectiveness of the proposed technique is examined by computing the approximate solution for various values of the α of the ultraspherical polynomial and compare wit the exact solution at $\mu = 2$, given by equation (2).

Table 1: Comparison of the errors at different values of $\alpha, \mu = 1.95, \delta t = \frac{1}{100}, \mathcal{N} = 3$

x	$\alpha = 0.5$	$\alpha = 1$	$5 \cdot \mathcal{N} = 4$
0.0	3.01×10^{-7}	2.33×10^{-7}	4.65×10^{-7}
0.2	5.44×10^{-7}	2.96×10^{-7}	7.65×10^{-7}
0.4	6.39×10^{-7}	4.20×10^{-7}	1.02×10^{-7}
0.6	8.45×10^{-7}	4.68×10^{-7}	1.99×10^{-6}
0.8	3.54×10^{-7}	2.63×10^{-7}	6.85×10^{-7}
1.0	3.08×10^{-9}	1.55×10^{-9}	2.15×10^{-9}

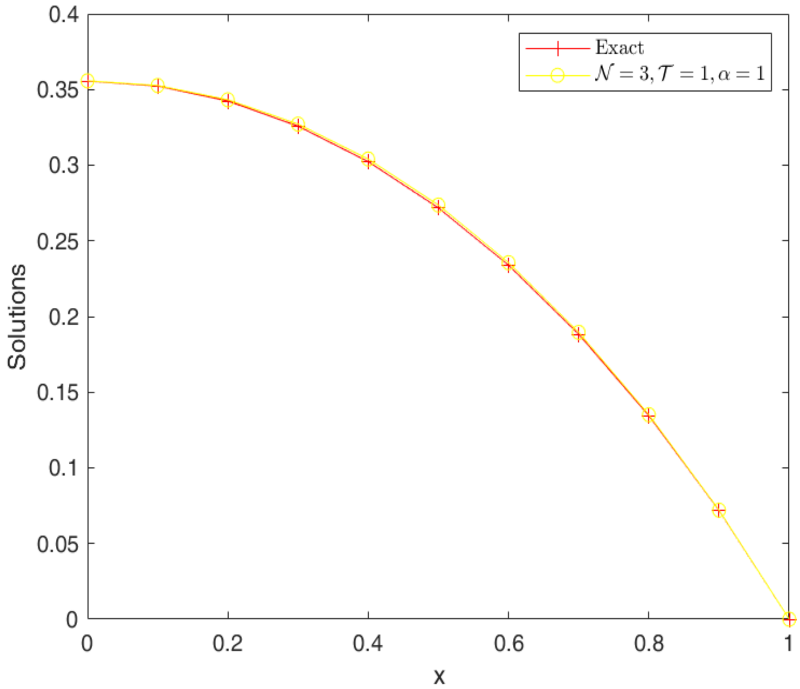

Figure 1: Exact solution and its corresponding $\mathcal{U}_{\mathcal{N}}(x, \mathcal{T} = 1), \mathcal{N} = 3$

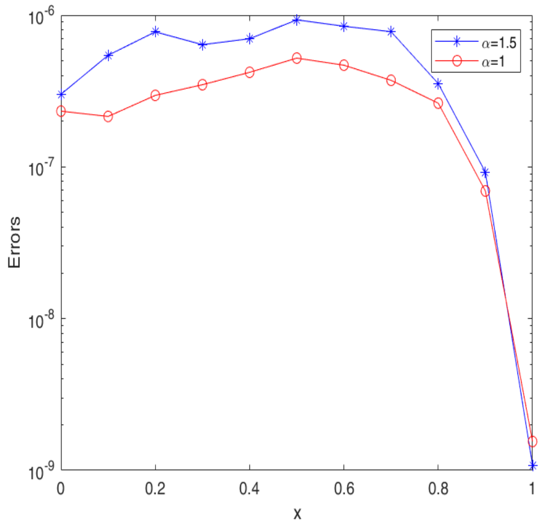

Figure 2: Comparison of the errors at various values of $\alpha, \mathcal{N} = 3, \mu = 1.85$

Conclusion

This paper presents an efficient numerical scheme for solving the Fisher-KPP equation based on the LC fractional derivative. The proposed scheme integrates compact finite difference strategies with shifted ultraspherical polynomials. We have demonstrated that the method is both unconditionally stable and convergent. To validate its effectiveness, several numerical experiments were conducted. The results, computed for various values of μ and α are compared with the exact solution. The numerical findings confirm that the scheme is robust and highly effective in solving the Fisher-KPP equation.

References

Abdeljawad, T., & Baleanu, D. (2017). Integration by parts and its applications of a new non-local fractional derivative with mittag-le_er nonsingular kernel. *The Journal of Nonlinear Sciences and Applications,* 10, 1098-1107. Available from http://dx.doi.org/10.22436/jnsa.010.03.20

Abdeljawad, T., & Baleanu, D. (2018). On fractional derivatives with generalized mittag-leffler kernels. *Adv. Differ. Equ.,* 468, s13662. Available from https://doi.org/10.1186/s13662-018-1914-2

Agarwal, F., & El-Sayed, A. A. (2020). Vieta-lucas polynomials for solving a fractional-order mathematical physics model. *Advances in Difference Equations,* 2020,. 626. Available from https://doi.org/10.1186/s13662-020-03085-y

Al-Qurashi, M. M., Korpinar, Z., Baleanu, D., & Inc, M. (2017). A new iterative algorithm on the time-fractional fisher equation: residual power series method. *Advances in Mechanical Engineering,* 9, 1-8. Available from DOI: 10.11771687814017716009

Alshammari, B. S., & Mashat, D. S. (2017). Numerical study of fisher's equation by finite difference schemes. *Applied Mathematics.*, 8, 1100-1116. Available from https://doi.org/10.4236/am.2017.88083

Atta, A. G., Moatimid, G. M., & Youssri, Y. H. (2020). Generalized fibonacci operational spectral tau algorithm for fractional bagley-torvik equation. *Progress in Fractional Differentiation and Applications,* 6, 1-10. Available from DOI:10 .18576/pfda/060305

Ağirseven, D., & Oziş, T. (2010). An analytic study for fisher type equations by using homotopy perturbation method. *Computers & Mathematics with Applications*, 60, 602-609.

Ervin, V. J., & Roop, J. P. (2007). Variational solution of fractional advection dispersion equations on bounded domains in $\mathbb{R}^d$. Numer. *Meth. Part. Diff. Eq.,* 23, 256-281. Available from DOI:10.1002/num. 20169

Issa, K., Yisa, B. M., & Biazar, J. (2022). Numerical solution of space fractional diffusion equation using shifted gegenbauer polynomials. Comput. *Methods for Differential Equations,* 10, 431- 444. Available from DOI:10.22034/cmde. 2020 .42106 .1818

Izadkhah, M. M., & Saberi-Nadja, J. (2015). Gegenbauer spectral method for timefractional convection-diffusion equations with variable coefficients. *Mathematical Methods in the Applied Sc.*, 38, 3183-3194. Available from DOI:10.1002/mma.3289

Kedia, N., Alikhanov, A. A., & Singh, V. K. (2024). Robust finite difference scheme for the non-linear generalized time-fractional diffusion equation with non-smooth solution. *Mathematics and Computers in Simulation,* 219, 337-354. Available from https://doi.org/10.1016/j.matcom.2023.12.034

Khader, M. M., & Adel, M. (2020). Numerical approach for solving the Riccati and logistic equations via qlm-rational Legendre collocation method. *Computational and Applied Mathematics,* 39 , 166. Available from https://doi.org/10.1007/s40314-020-01207-6

Kumar, S., Kumar, K., Pandey, R. K., & Xu, Y. (2024). Legendre collocation method for new generalized fractional advection-diffusion equation. *Intl. J. Comput. Math.,* 101, 1-23. Available from https://doi.org/10.1080/00207160.2024.2305640

Mabrouk, S. M., Rezazadeh, H., Ahmad, H., Rashed, A. S., Demirbilek, U., & Khaled, A. (2024). Implementation of optical soliton behavior of the space-time conformable fractional vakhnenko parkes equation and its modified model. *Opt. Quant. Electron,* 56, 222. Available from https://doi.org/10.1007/s11082-023-055537

Nagy, A. M., & Issa, K. (2024). An accurate numerical technique for solving fractional advection-diffusion equation with generalized caputo derivative. *ZAMP.,* 75, 164. Available from https://doi.org/10.1007/s00033-024-02309-5

Newman, W. I. (1980). Some exact solutions to a non-linear diffusion problem in population genetics and combustion*. Journal of Theoretical Biology,* 85, 325-334. Available from https://doi.org/10.1016/0022-5193(80)90024-7

Oruç, Ö (2020). An efficient wavelet collocation method for nonlinear two-space dimensional fisher- kolmogorov-petrovsky-piscounov equation and two-space dimensional extended fisher-kolmogorov equation. *Engineering with Computers,* 36, 839856. Available from https://doi.org/10.1007/S00366-019-00734-Z

Podlubny, I. (1999). Fractional differential equations. London, UK. Sweilam, N. H., Ahmed, S. M., & Adel, M. (2020). A simple numerical method for twodimensional nonlinear fractional anomalous sub- diffusion equations. *Mathematical Methods in the Applied Sciences,* 12, 1-20. Available from https://doi.org/ 10.1002/mma.6149

Sweilam, N. H., Nagy, A. M., & Al-Ajami, T. M. (2021). Numerical solutions of fractional optimal control with caputo-katugampola derivative. *Adv. Differ. Equ.*, 2021, 425. Available from https://doi.org/10.1186/s13662-021-03580-w

Szegö, G. (1975). Orthogonal polynomials. 4th Ed., AMS Colloq. Publ. Youssef, M. Z., Khader, M. M., Al-Dayel, I., & Ahmed, W. E. (2022). Solving fractional generalized Fisher-Kolmogorov-Petrovsky-Piskunov's equation using compact- finite different methods together with spectral collocation algorithms. *Journal of Mathematics,* 2022, 1901131. Available from https://doi.org/ 10.1155/2022/1901131

Zhang, X., & Liu, J. (2014). An analytic study on time-fractional fisher equation using homotopy perturbation method. *Walailak Journal of Science and Technology,* 11, 975-985. Available from https://doi.org/10.14456/WJST.2014.72