ON THE SEMIGROUP RING OF THE RHOTRIX BICYCLIC SEMIGROUP

***AHMED, B. M., BAKARE, G. N., & USAMOT, I. F.**
Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Nigeria
***Email:** ahmed.bm@unilorin.edu.ng ***Phone No:** +234-8035233619

Abstract

In this paper, we study the algebraic structure of the semigroup ring associated with the rhotrix bicyclic semigroup $B(N_0)$*. Let* R *be a ring and* $B(N_0)$ *the rhotrix bicyclic semigroup defined on ordered pairs of sequences indexed by a fixed index set. We construct the semigroup ring* $R[B(N_0)]$ *and investigate some of its structural properties. In particular, we establish that* $R[B(N_0)]$ *is generally non-commutative and determine a family of idempotent elements arising from idempotents of* $B(N_0)$*. The ordering of these idempotents is described componentwise, and this ordering induces a natural hierarchy within the semigroup ring. Furthermore, several classes of subrings of* $R[B(N_0)]$ *are identified, including subrings generated by subsemigroups, idempotents, and corner subrings determined by idempotent elements. These results reveal how the structural properties of the rhotrix bicyclic semigroup influence the internal structure of its semigroup ring and provide a foundation for further study of ideals and related algebraic properties.*

Keywords: Corner subrings; idempotent elements; ordered idempotents; rhotrix bicyclic semigroup; semigroup ring; subrings.

Introduction

According to Howie (1976), a semigroup is a nonempty set equipped with binary operation that is closed and associative. Semigroup theory has developed into an important branch of algebra due to its wide range of applications and its strong connections with other algebraic structures such as groups, rings and monoids.

The concept of rhotrix was introduced by Ajibade (2003) as an extension of the ideas of matrix tertions and matrix noitrets earlier proposed by Atanassov and Shannon [3]. Since its introduction, the theory of rhotrices has attracted considerable attention from researchers who have explored various algebraic structures built on rhotrix sets.

For instance, Mohammed, Balarabe and Imam (1998) presented a method of constructing certain semigroups using rhotrix sets as the underlying sets together with the binary operation for rhotrix multiplication proposed by Sani (2007). In another direction, Tudunkaya (2013) investigated the properties and nature of polynomials defined over ring of rhotrices and rhotrices defined over polynomial rings. Furthermore, Mohammed and Shettima (2017) studied certain properties of rings consisting of rhotrices of size three over the field of real numbers and examined their subrings.

More recently, Isere (2018) introduced even-dimensional rhotrices, extending the classical notion where rhotrices of odd dimensions were predominantly considered in the literature. Even-dimensional rhotrices arise when the central "heart" element removed, and several structural properties of these rhotrices were established. In addition, Isere (2016) also studied the set of rhotrices whose entries are ordered natural numbers, referred to as natural rhotrices, and examined several algebraic properties of this set.

Semigroup rings constitute an important area of study in algebra because they provide a natural way of combining the structure of rings and semigroups. Given a ring R and a semigroup S, the semigroup ring $R[S]$ allows algebraic techniques from ring theory to be applied in the study of semigroups. The structural properties of semigroup rings, such as idempotents, ideals, and subrings are often closely related to the internal structure of the underlying semigroup.

The bicyclic semigroup is one of the most fundamental examples in semigroup theory and has been widely studied due to its rich structural properties, particularly in relation to idempotents and Green's relations. In particular, certain subgroups of the bicyclic semigroup were previously investigated by Makanjuola and Umar (1997), whose work provides motivation for further investigations in related algebraic constructions.

Motivated by these developments, the concept of the rhotrix bicyclic semigroup was recently introduced in the author's earlier work titled "The Concept of Rhotrix Bicyclic Semigroup". This present paper continues this line of research by studying the semigroup ring generated by this semigroup. In particular, we construct the semigroup ring $R[B(\mathbb{N}_0)]$ associated with the rhotrix bicyclic semigroup and investigate several of its structural properties, including idempotent elements, ordering of idempotents, and various classes of subrings generated within the ring.

The results obtained in this work further illuminate the interaction between rhotrix-based semigroup structures and their associated semigroup rings and provide a foundation for further investigations on ideals and related algebraic properties.

A rhotrix is a rhomboidal array of real numbers whose structure lies between the (2×2) and (3×3) matrix configurations. Thus,

$$M=\left\{\left\langle\begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array}\right\rangle : a,b,c,d,e\in\mathbb{R}\right\} \tag{1}$$

Where c is the heart of the rhotrix usually denoted as $h(M)$. The operations of addition and multiplication for heart-based rhotrices were first introduced by Ajibade (2003) and are defined as follows.

$$M+N=\left\langle\begin{array}{ccc} & a & \\ b & h(M) & d \\ & e & \end{array}\right\rangle+\left\langle\begin{array}{ccc} & f & \\ g & h(N) & j \\ & k & \end{array}\right\rangle=\left\langle\begin{array}{ccc} & a+f & \\ b+g & h(M)+h(N) & d+j \\ & e+k & \end{array}\right\rangle \tag{2}$$

And

$$M\circ N=\left\langle\begin{array}{ccc} & ah(N)+fh(M) & \\ bh(N)+gh(M) & h(M)h(N) & dh(N)+jh(M) \\ & eh(N)+kh(M) & \end{array}\right\rangle \tag{3}$$

respectively. A generalization of the product was later presented by Mohammed (2011). An alternative multiplication method for heart based rhotrices, known as "row-column" method, was introduced by Sani (2004) as

$$M \circ N = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(M)h(N) & aj+dk \\ & bj+ek & \end{array} \right\rangle \tag{4}$$

A generalization of this row-column multiplication was also given by Sani (2007).

Preliminary

The following definitions will serve as a foundation for the discussion in the subsequent sections.

Definition 2.1 (Rhotrix Bicyclic Semigroup)

Let $B(N_0)$ denote the rhotrix bicyclic semigroup denoted by

$$B(N_0) := \{(M,N) : M = (m_i)_{i \in I}, N = (n_i)_{i \in I}, \; m_i, n_i \in N_0 \}.$$

Where:

(i) I is a fixed set

(ii) M and N are sequence indexed by I

Each component m_i, n_i belongs to N_0.

Multiplication is defined componentwise by $(M,N)*(R,S)=(P,Q)$
where for each $i \in I$, $p_i = m_i + r_i - \min(n_i, r_i)$, $q_i = n_i + s_i - \min(n_i, r_i)$

Definition 2.2 (Semigroup Ring)

Let R be a ring and let S be a semigroup. The semigroup ring of S over R, denoted by $R[S]$, is the set of all finite formal sums

$$\sum_{i=1}^{k} a_i s_i$$

where $a_i \in R$, $s_i \in S$ and k is a finite integer.

Definition 2.3 (Semigroup Ring of the Rhotrix Bicyclic Semigroup)

Let R be a ring and let $B(N_0)$ be the rhotrix bicyclic semigroup. The semigroup ring of the rhotrix bicyclic semigroup over R is the ring $R[B(N_0)]$ consisting of all finite formal sums

$$\sum_{k=1}^{t} a_k (M_k, N_k)$$

where $a_k \in R$ and $(M_k, N_k) \in B(N_0)$

Definition 2.4 (Addition on $R[B(N_0)]$)

Let R be a ring and let $B(N_0)$ be the rhotrix bicyclic semigroup. Consider two elements of the semigroup ring $R[B(N_0)]$

$$x = \sum_{i=1}^{k} a_i (M_i, N_i), \qquad y = \sum_{i=1}^{k} b_i (M_i, N_i)$$

where $a_i, b_i \in R$ and $(M_i, N_i) \in B(N_0)$

The addition on $R[B(N_0)]$ is defined by

$$x + y = \sum_{i=1}^{k} (a_i + b_i)(M_i, N_i)$$

Thus the coefficients are added in the ring R, while the semigroup elements remain unchanged.

Remark 2.1

Addition in $R[B(N_0)]$ is therefore coefficientwise, inherited from the addition in coefficient ring R.

Definition 2.5 (Multiplication on $R[B(N_0)]$)

Let R be a ring and let $B(N_0)$ be the rhotrix bicyclic semigroup. Consider two elements of the semigroup ring $R[B(N_0)]$

$$x = \sum_{i=1}^{k} a_i (M_i, N_i), \qquad y = \sum_{j=1}^{t} b_j (R_j, S_j)$$

Where $a_i, b_j \in R$ and $(M_i, N_i)*(R_j, S_j) \in B(N_0)$

The multiplication on $R[B(N_0)]$ is defined by extending the multiplication in $B(N_0)$ linearly as follows:

$$xy = \sum_{i=1}^{k} \sum_{j=1}^{t} a_i b_j ((M_i,N_i)*(R_j,S_j))$$

Here the product $(M_i,N_i)*(R_j,S_j)$ is computed using the multiplication defined in the rhotrix bicyclic semigroup $B(N_0)$.

Remark 2.2

The multiplication in $R[B(N_0)]$ is therefore obtained by combining the multiplication in the coefficient ring R with the semigroup multiplication in $B(N_0)$ and extending distributively.

Definition 2.6 (Ordering of Idempotents)

Let $e=I_R(M,M)$, $f=I_R(N,N)$ be idempotents in $R[B(N_0)]$. We define $e \leq f$ if $ef=fe=e$.

Definition 2.7 (Subring)

Let A be a ring. A subset $S \subseteq A$ is called a subring of A if:

(i) S is closed under addition and multiplication

(ii) S contains the additive identity of A

(iii) S contains the additive inverse of each of its elements.

Definition 2.8 (Corner Subring)

Let A be a ring and e be an idempotentin A. The set

$$eAe = \{exe : x \in A\}$$

Is called the corner subring of A determine by e.

Main Results

Theorem 3.1

Let R be a ring and let $B(N_0)$ be rhotrix bicyclic semigroup. Then the semigroup ring n $R[B(N_0)]$ forms a ring under the addition and multiplication defined above.

Proof

First, we show that n $R[B(N_0)]$ is an abelian group under addition.

Let $x=\sum a_i(M_i, N_i)$, $y=\sum b_i(M_i, N_i)$ be any two elements in $R[B(N_0)]$. Since addition in R is associative and commutative, it follows that addition in $R[B(N_0)]$, defined by

$$x+y=\sum (a_i+b_i)(M_i, N_i)$$

is also associative and commutative.

The element $0=\sum 0(M, N)$ serves as the additive identity,

and for every $x=\sum a_i(M_i, N_i)$, the element $-x=\sum (-a_i)(M_i, N_i)$ is its additive inverse.

Hence $R[B(N_0)]$ is an abelian group under addition.

Next, multiplication in $R[B(N_0)]$ is defined by

$$\left(\sum a_i(M_i, N_i)\right)\left(\sum b_j(R_j, S_j)\right)=\sum_{ij} a_i b_j((M_i, N_i)*(R_j, S_j)).$$

Since multiplication in R is associative and the multiplication in $B(N_0)$ is associative, it follows that multiplication in $R[B(N_0)]$ is associative.

Finally, the distributive laws

$$x(y+z)=xy+xz \quad \text{and} \quad (x+y)z=xz+yz$$

hold because multiplication is defined distributively over addition.

Therefore, $R[B(N_0)]$ satisfies all the axioms of a ring.

Lemma 3.1

The rhotrix bicyclic semigroup $B(N_0)$ has an identity element.

Proof

Let $M_0=(0)_{i\in I}$, $N_0=(0)_{i\in I}$,

That is, M_0 and N_0 are sequences whose componenets are all zero.

Consider any element $(M,N)\in B(N_0)$ where $M=(m_i)_{i\in I}$, $N=(n_i)_{i\in I}$

Using the multiplication in $B(N_0)$,

$(M_0, N_0)(M,N)=(P,Q)$, where for each $i\in I$

$p_i=0+m_i-\min(0,m_i)$, $q_i=0+n_i-\min(0,m_i)$

Since $\min(0,m_i)=0$, we obtain $p_i=m_i$, $q_i=n_i$.

Thus $(M_0, N_0)(M,N)=(M,N)$. Similarly, $(M,N)(M_0, N_0)=(M,N)$.

Hence, (M_0, N_0) is the identity element of $B(N_0)$.

Proposition 3.1

If R is a ring with identity I_R, then the semigroup ring $R[B(N_0)]$ has identity $I_R(M_0.N_0)$, where $(M_0.N_0)$ is the identity element of $B(N_0)$.

Proof

Let $x=\sum a_i(M_i, N_i)\in R[B(N_0)]$. Then

$$I_R(M_0.N_0)\,.\,x=\sum I_R a_i((M_0.N_0)*(M_i, N_i))$$

Since $(M_0.N_0)$ is the identity of $B(N_0)$,

$(M_0.N_0)*(M_i,N_i) = (M_i,N_i)$. Thus

$I_R(M_0.N_0)\,.\,x=\sum a_i(M_i, N_i)=x$.

Similarly, $x\,.\,I_R(M_0.N_0)=x$

Therefore, $I_R(M_0.N_0)$ is the identity element of $R[B(N_0)]$.

Theorem 3.2

If R is a commutative ring with identity, then the semigroup ring $R[B(N_0)]$ is generally non-commutative.

Proof

It suffices to show that the multiplication in $B(N_0)$ is not commutative.

Let $(M,N), (R,S)\in B(N_0)$. Then

$(M,N)*(R,S)=(P,Q)$

where $p_i=m_i+r_i-\min(n_i, r_i)$, $q_i=n_i+s_i-\min(n_i, r_i)$

Similarly,

$(R,S)*(M,N)=(P',Q')$

Where $p_i'=r_i+m_i-\min(s_i,m_i)$, $q_i'=s_i+n_i-\min(s_i,m_i)$

Since in general, $\min(n_i, r_i)\neq \min(s_i,m_i)$.

It follows that $(P,Q)\neq (P',Q')$

Hence, $(M,N)*(R,S)\neq (R,S)*(M,N)$

Thus $B(N)$ forms the multiplicative basis of the semigroup ring $R[B(N_0)]$, the ring multiplication inherits this non-commutativity. Therefore, $R[B(N_0)]$ is not commutative in general.

Idempotents in $B(N_0)$ and $R[B(N_0)]$

Proposition 3.2
For every sequence $M=(m_i)_{i\in I}$, the element $(M,M)\in B(N_0)$ is an idempotent element.

Proof
Let $(M,M)\in B(N_0)$ where $M=(m_i)_{i\in I}$
Using multiplication defined in $B(N_0)$,

$$(M,M)*(M,M)=(P,Q)$$

Where for each $i\in I$

$$p_i=m_i+m_i-\min(m_i,m_i), \qquad q_i=m_i+m_i-\min(m_i,m_i)$$

Since $\min(m_i,m_i)=m_i$, we obtain $p_i=m_i$, $q_i=m_i$
Hence,

$$(M,M)*(M,M)=(M,M)$$

Therefore, (M,M) is an idempotent element of $B(N_0)$.

Proposition 3.3
If R is a ring with identity, then the element $I_R(M,M)$ is an idempotent in the semigroup ring $R[B(N_0)]$.

Proof
Since (M,M) is an idempotent in $B(N_0)$,

$$(M,M)^2=(M,M).$$

Thus in the semigroup ring

$$(I_R(M,M))^2=I_R{}^2(M,M)^2$$

Because $I_R{}^2=I_R$, we obtain

$$(I_R(M,M))^2=I_R(M,M).$$

Hence $I_R(M,M)$ is an idempotent in $R[B(N_0)]$

Remark 2.3
The set $E=\{(M,M):\ M=(m_i)_{i\in I}\}$ forms a family of idempotents in the rhotrix bicyclic semigroup $B(N_0)$. Consequently, the semigroup ring $R[B(N_0)]$ contains infinitely many idempotent elements arising from these semigroup idempotents.

Ordering of Idempotents in $R[B(N)]$
Let $E(B(N_0))=\{(M,M):\ M=(m_i)_{i\in I}\}$ denote the set of idempotents of the rhotrix bicyclic semigroup $B(N)$.
Since each element (M,M) appears in the semigroup ring $R[B(N_0)]$, these idempotent naturally occur in the ring as $I_R(M,M)$.

Theorem 3.3
Let $e=I_R(M,M)$, $f=I_R(R,R)$ be idempotents in $R[B(N_0)]$. Then $e\leq f$ if and only if $m_i\leq r_i$ for all $i\in I$.

Proof

$$ef=I_R{}^2(M,M)*(R,R). \quad \text{Since } I_R{}^2=I_R$$

$$ef=I_R((M,M)*(R,R))$$

From the multiplication in $B(N_0)$,

$$(M,M)*(R,R)=(P,P)$$

Where $p_i=\max(m_i,r_i)$
Thus $ef=I_R(P,P)$
For $ef=e$, we must have $P=M$
Hence, $\max(m_i,r_i)=m_i$

This occurs exactly when $m_i \leq r_i$ for all $i \in I$. Thus the ordering of idempotents corresponds to the componentwise ordering of the sequences.

Corollary 3.1
The set of idempotents of $R[B(N_0)]$

$E(R[B(N_0)]) = \{I_R(M,M): M=(m_i)_{i\in I}\}$ forms partially ordered set under the above relation.

Subrings of the Semigroup Ring R[B(N_0)]

Let R be a ring and let $B(N_0)$ be the rhotrix bicyclic semigroup. Consider the semigroup ring $R[B(N_0)]$. In this section we describe some natural subrings of $R[B(N_0)]$.

Proposition 4.1 (Coefficient Ring as a Subring)
Let $T = \{a(M_0,M_0): a \in R\}$
Where (M_0,M_0) is the identity element of $B(N_0)$. Then T is a subring of $R[B(N_0)]$.

Proof
Let $a(M_0,M_0)$, $b(M_0,M_0) \in T$. Then

$$a(M_0,M_0) + b(M_0,M_0) = (a+b)(M_0,M_0) \in T.$$

Also,

$$a(M_0,M_0) \,.\, b(M_0,M_0) = ab((M_0,M_0)(M_0,M_0))$$

Since (M_0,M_0) is the identity of $B(N_0)$,

$$(M_0,M_0) * (M_0,M_0) = (M_0,M_0).$$

Thus,

$$a(M_0,M_0) \,.\, b(M_0,M_0) = ab(M_0,M_0) \in T.$$

The additive identity $0(M_0,M_0)$ belongs to T, and additive inverse also lie in T. Hence, T is a subring of $R[B(N_0)]$.

Proposition 4.2 (Subrings Generated by Subsemigroups)
Let S be a semigroup of $B(N_0)$. Then

$R[S] = \{\sum a_i s_i: a_i \in R, s_i \in S\}$ is a subring of $R[B(N_0)]$.

Proof
Since S is closed under multiplication, the product of any two elements of $R[S]$ remain in $R[S]$. Also, sums and additive inverses remains in $R[S]$ because they are defined using coefficient from R. Hence, $R[S]$ satisfies all the conditions of a subring.

Example 4.1
Consider $S = \{(M,O): M=(m_i)_{i\in I}, m_i \in N\}$. Then

$R[S] = \{\sum a_i(m_i, 0)\}$ is a subring.

Proposition 4.3 (Subring Generated by Idempotents)
Let $E = \{(M,M): M=(m_i)_{i\in I}\}$ be the set of idempotents of $B(N_0)$. Then the set

$$R[E] = \{\sum a_i(M_i, M_i): a_i \in R\}$$

forms a subring of $R[B(N_0)]$.

Proof
Since the product of two idempotents in $B(N_0)$ is again an idempotent, the set E is closed under multiplication. Hence, $R[E]$ is closed under addition and multiplication, and therefore forms a subring of $R[B(N_0)]$.

Remark 2.4
These subrings illustrate how the structure of the rhotrix bicyclic semigroup influences the internal structure of the semigroup ring $R[B(N_0)]$. In particular, subsemigroups and idempotents of $B(N_0)$ naturally generate subrings of $R[B(N_0)]$.

Chain of Subrings Generated by Idempotents
Recall that the set of idempotents of $B(N_0)$ is

$$E=\{(M,M):\ (m_i)_{i\in I}\}$$

For each idempotent (M,M), consider the subset

$$R_M=\{\sum a_j(M_j,M_j):(M_j,M_j)\leq(M,M),\ a_j\in R\}$$

Theorem 4.1
Let $(M,M),(R,R)\in E$
If $(M,M)\leq(R,R)$ then $R_M\subseteq R_R$
Hence the family R_M forms a chain of subrings of $R[B(N_0)]$.

Proof
Suppose $(M,M)\leq(R,R)$.
By the ordering idempotents in $B(N_0)$, this means

$$m_i\leq r_i \quad \text{for all} \quad i\in I.$$

Let $x\in R_M$. Then $x=\sum a_j(M_j,M_j)$ where $(M_j,M_j)\leq(M,M)$
Since $(M,M)\leq(R,R)$, it follows that $(M_j,M_j)\leq(R,R)$.
Thus every term appearing in x also belongs to R_R.
Hence $x\in R_R$. Therefore, $R_M\subseteq R_R$.
Thus the subrings generated by idempotents form an ordered chain.

Corollary 4.1
The semigroup ring $R[B(N_0)]$ contains infinitely many nested subrings generated by the idempotents of the rhotrix bicyclic semigroup.

Remark 2.5
The chain of subrings induced by idempotents reflects the ordered structure of the idempotents set of $B(N_0)$.

Corner Subrings Generated by Idempotents
Let $R[B(N_0)]$ be the semigroup ring of the rhotrix bicyclic semigroup and let

$e=I_R(M,M)$ be an idempotent in $R[B(N_0)]$

Theorem 4.2
Let $e=I_R(M,M)$ be an idempotent of $R[B(N_0)]$. Then $eR[B(N_0)]e$ is a subring of $R[B(N_0)]$.

Proof
Let $x,y\in eR[B(N_0)]e$. Then there exist $a,b\in R[B(N_0)]$ such that

$$x=eae, \qquad y=ebe$$

Closure under addition:

$$x+y=eae+ebe=e(a+b)e$$

Since $a+b\in R[B(N_0)]$, it follows that
$x+y\in e\ R[B(N_0)]e$
Closure under multiplication:
$xy=(eae)(ebe)$. Using the idempotent property $e^2=e$,

$$xy=eaebe=e(aeb)e$$

Since $aeb \in R[B(N_0)]$, we obtain $xy \in eR[B(N_0)]e$.
Additive identity:
Taking $x=0$, we have $e0e=0$
So the additive identity belongs to $eR[B(N_0)]e$
Additive inverses:
If $x=eae$, then $-x=e(-a)e$, which also belongs to $eR[B(N_0)]e$.
Thus all subring conditions hold, and therefore $eR[B(N_0)]e$ is a subring of $R[B(N_0)]$.

Remark 2.6
Each idempotent (M,M) in the rhotrix bicyclic semigroup generates a corresponding corner subring in the semigroup ring $R[B(N_0)]$. Consequently, the internal structure of $R[B(N_0)]$ contains many such subrings determined by the idempotent elements of $B(N_0)$.

Conclusion

In this paper, the semigroup ring associated with the rhotrix bicyclic semigroup was constructed and its structural properties were investigated. In particular, the semigroup ring $R[B(N_0)]$ was defined and shown to inherit important algebraic features from the underlying rhotrix bicyclic semigroup. It was observed that the ring is generally non-commutative, and a family of idempotent elements arising from the idempotents of $B(N_0)$ was identified.

Furthermore, the ordering of these idempotents was examined and shown to correspond to the componentwise ordering of the sequences defining the elements of the rhotrix bicyclic semigroup. This ordering provides additional insight into the internal structure of the semigroup ring. In addition, several classes of subrings of $R[B(N_0)]$ were described, including those generated by subsemigroups and idempotent elements.

The results obtained in this work demonstrate how the algebraic properties of the rhotrix bicyclic semigroup influence the structure of its associated semigroup ring. These findings provide a foundation for further investigations on ideals, Green's relations, and other structural properties of $R[B(N_0)]$.

References

Ahmed, B. M., Yisa, B. M., & Ayinla, Y. A. (2025). On subgroups of non-commutative orthogonal rhotrix group. *Al-hikmah Journal of Pure and Applied Sciences,* 5(1), 1 – 11.

Ajibade, A. O. (2003). The concept of rhotrix in mathematical enrichment. *International Journal of Mathematical Education in Science and Technology,* 34(2), 175 – 179.

Atanassov, K. T., & Shannon, A. G. (1998). Matrix-tertions and matrix-noitrets: Exercises in mathematical enrichment. *International Journal of Mathematical Education in Science and Technology,* 29(6), 898-903.

Howie, J. M. (1976). *An introduction to semigroup theory.* London Mathematical Society (monograph)

Howie, J. M. (1995). *Fundamental of semigroup theory.* Oxford University Press, USA

Isere, A. O. (2016). Natural numbers. *Cogent Mathematics*, 3(1), 1 – 10.

Isere, A. O. (2018). Even dimension rhotrix. Notes on Number theory and Discrete. *Mathematics,* 24(2), 125 – 133.

Makanjuola, S. O., & Umar, A. (1997). On a certain subsemigroup of bicyclic semigroup. *Communication in Algebra,* 25(2), 509 – 519.

Mohammed, A. (2007). Enrichment exercises through extension to rhotrices. *International Journal of Mathematical Education in Science and Technology,* 38(1), 131-136.

Mohammed, A. (2009). A remark on the classification of rhotrices as abstract structures. *International Journal of Physical Sciences,* 4(9), 496-499.

Mohammed, A. (2011). Theoretical development and application of rhotrices. Ph. D. Thesis, Ahmadu Bello University, Zaria.

Mohammed, A. (2018). The non-commutative full rhotrix ring and its subrings. *Science World Journal,* 13(2), 24 – 36.

Mohammed, A., Balarabe, M., & Imam, A. (2014). On construction of rhotrix semigroup. *Nigerian Association of Mathematical Physics,* 27, 69-76.

Mohammed, A., & Okon, U. E. (2015). On subgroups of non-commutative general rhotrix group. *Journal of the Nigerian Association of the Mathematical Physics,* 30, 49-64.

Mohammed, A., & Shettima, M. (2017). On subrings of commutative ring of all real rhotrices of size three. *Journal of the Nigerian Association of Mathematical Physics,* 43, 19 – 30.

Malik, D. S., Mordeson, J. N., & Sen, M. K. (2007). *Introduction to Abstract Algebra.* Cengage Learning.

Sani, B. (2004). An alternative method for multiplication of rhotrices. *International Journal of Mathematical Education in Science and Technology,* 35(5), 777-781.

Sani, B. (2007). The row-column multiplication of high dimensional rhotrices. *International Journal of Mathematical Education in Science and Technology,* 38(5), 657-662.

Tudunkaya, S. M. (2013). Rhotrix polynomials and polynomial rhotrices. *Pure and Applied Mathematics Journal,* 2(1), 38 – 41.