NUMERICAL SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND USING CERTAIN QUADRATURES

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Abstract
In this paper, nonhomogeneous Fredholm integral equations of the second kind are approximated using Boole's and Weddle's rules. The kernels of the family of integral equations are approximated by the two quadrature formulae due to their accuracy. In most of the problems solved, the two quadratures gives exact solutions, but where the exact solution is not obtained, the numerical approximation realized are very reasonable.

Keywords: Integral Equations, Quadrature, Fredholm, Boole's Rule, Weddle's Rule

Introduction
The Fredholm integral equation of the form

$$U(x)+\lambda\int_a^b K(x,t)U(t)dt = f(x), \qquad a \le x,t \le b \tag{1}$$

Where $U(x)$ is the unknown function that appears both under the integral sign and outside it, thus making (1) Fredholm integral equation of the second kind. λ is a constant with $|\lambda| = 1$, $K(x,t)$ is a function of two variables x and t, while λ is called the kernel (or nucleus) of the integral equation. $K(x,t)$ is a smooth function. $f(x)$ is the nonhomogeneous source term which is a function of x.
The class of problems represented by (1), occur most frequently in many branches of engineering and mathematical physics. For instance in electrostatics as represented in Love's equation (Love, 1949) and (Curtis, 1960), and was improved upon by (El – Gendi, 1969). Several other approaches were adopted by various researchers to handle problems stated in (1). Prominent among such approaches are the Chebyshev series method due to (Elliot, 1963), Chebyshev iteration method used by (Sag, 1970), Chebyshev and Legendre collocation method due to (Ghadle and Ahmed, 2016) and many other ones highlighted in the integral equations of (wazawaz, 2011). In this work, (1) is studied using two numerical quadratures, viz: Boole's and Wedle's rules.

Boole's and Weddle's Rules
The two quadratures used in this work are stated here for the sake of completeness.

Boole's rule
Consider the approximation of $\int_{x_0}^{x_n} U(x)dx$ by taking the curve representing the function,

$U(x)$, through four points. Since $x \in [x_0, x_n]$, when partioned into n sub – intervals, the step size is given by $h = \frac{x_n - x_0}{n}$. Thus, the Boole's method will be obtained as

$$\int_{x_0}^{x_n} U(x)dx = \frac{2h}{45}[7U(x_0) + 32[U(x_1)+U(x_3)+U(x_5)+...] + 12[U(x_2)+U(x_6)+U(x_{10})+...]$$
$$+14[U(x_4)+U(x_8)+U(x_{12})+...] + 7U(x_n)] \tag{2}$$

Weddle's Rule
On the other hand, if the curve is taken through six points at a time, and by that covering the entire interval $x_0 \le x \le x_n$, we obtain Weddle's rule as

$$\int_{x_0}^{x_n} U(x)dx = \frac{3h}{10}[U(x_0)+5[U(x_1)+U(x_5)+U(x_7)+U(x_{11})...]+[U(x_2)+U(x_4)+U(x_8)+U(x_{10})+...]$$
$$+6[U(x_3)+U(x_9)+U(x_{15})+...]+2[U(x_6)+U(x_{12})+U(x_{18})+...]+U(x_n)] \quad (3)$$

Statement of Problem
The problem considered in this paper is the Fredholm integral equation of the second kind given as

$$U(x)+\int_a^b K(x,t)U(t)dt = f(x), \qquad a \le x,\ t \le b, \quad (4)$$

with $|\lambda|=1$, a and b as arbitrary constants, the kernel, $K(x,t)$, as a smooth function of two variables, $f(x)$ is a function of x (that is majorly a polynomial or a function that can be approximated by Taylor's series) and $U(x)$ is the unknown function.

Theorem (Uniqueness of Solution) (Wazawaz, 2011)
If the kernel, $K(x,t)$, in Fredholm integral equation (4) is continuous, bounded in the square $a \le x \le b$ and $a \le t \le b$, and if $f(x)$ is a continuous real valued function, the necessary condition for the existence of a unique solution for the Fredholm integral equation (4) is given by

$$|\lambda| M(b-a) < 1, \quad (5)$$

and

$$|K(x,t)| \le M \in R \quad (6)$$

Methodology
In solving (4), the kernel is approximated by using (2) and (3) as illustrated in the sequel.

Application of Boole's Rule
If the integrand in (4) can be expressed as

$$\int_a^b K(x,t)U(t)dt = \sum_{\lambda=0}^{N} A_\lambda K(x,t)U(t_\lambda), \quad (7)$$

With (7), (4) can be written as

$$U(x)+\frac{2h}{45}[7K(x,t_0)U(t_0)+32[K(x,t_1)U(t_1)+K(x,t_3)U(t_3)+K(x,t_5)U(t_5)+...]$$
$$+12[K(x,t_2)U(t_2)+K(x,t_6)U(t_6)+K(x,t_{10})U(t_{10})+...]$$
$$+14[K(x,t_4)U(t_4)+K(x,t_8)U(t_8)+K(x,t_{12})U(t_{12})+...]+7K(x,t_n)U(t_n)] = f(x) \quad (8)$$

Next, the variable x in (8) is replaced by $t_i, i=0(1)n,$ and $U(t_i)$ is taken to be U_i, thus (8) is to be rewritten as

$$U_i + \frac{2h}{45}[7K(t_i,t_0)U_0 + 32[K(t_i,t_1)U_1 + K(t_i,t_3)U_3 + K(t_i,t_5)U_5 + ...]$$
$$+12[K(t_i,t_2)U_2 + K(t_i,t_6)U_6 + K(t_i,t_{10})U_{10} + ...] + 14[K(t_i,t_4)U_4 + K(t_i,t_8)U_8 + K(t_i,t_{12})U_{12} + ...]$$
$$+7K(t_i,t_n)U_n] = f(x), \quad i = 0(1)n \tag{9}$$

(9) will generate (n+1) equations in (n+1) unknowns. The equations thus generated can be solved using any of the linear equation solvers, such as, Mathematica, Maple, Mathcard. The values of the unknowns are substituted back in (8) to get the required $U(x)$.

Application of Weddle's Rule

The Weddle's rule stated in (3) is also implemented in (4) by rewriting it in the form

$$U(x) + \frac{3h}{10}[K(x,t_0)U(x_0) + 5[K(x,t_1)U(x_1) + K(x,t_5)U(x_5) + K(x,t_7)U(x_7) + K(x,t_{11})U(x_{11}) + ...]$$
$$+[K(x,t_2)U(x_2) + K(x,t_4)U(x_4) + K(x,t_8)U(x_8) + K(x,t_{10})U(x_{10}) + ...]$$
$$+6[K(x,t_3)U(x_3) + K(x,t_9)U(x_9) + K(x,t_{15})U(x_{15}) + ...]$$
$$+2[K(x,t_6)U(x_6) + K(x,t_{12})U(x_{12}) + K(x,t_{18})U(x_{18}) + ...] + K(x,t_n)U(x_n)] = f(x) \tag{10}$$

Also replacing x by t_i and taking $U(t_i)$ as U_i, (10) can be rewritten as

$$U_i + \frac{3h}{10}[K(t_i,t_0)U_0 + 5[K(t_i,t_1)U_1 + K(t_i,t_5)U_5 + K(t_i,t_7)U_7 + K(t_i,t_{11})U_{11} + ...]$$
$$+[K(t_i,t_2)U_2 + K(t_i,t_4)U_4 + K(t_i,t_8)U_8 + K(t_i,t_{10})U_{10} + ...]$$
$$+6[K(t_i,t_3)U_3 + K(t_i,t_9)U_9 + K(t_i,t_{15})U_{15} + ...]$$
$$+2[K(t_i,t_6)U_6 + K(t_i,t_{12})U_{12} + K(t_i,t_{18})U_{18} + ...] + K(t_i,t_n)U_n] = f_i, \qquad i = 0(1)n \tag{11}$$

(11) generates (n+1) equations in (n+1) unknowns. The values of the unknown constants $U_0, U_1, U_2, ..., U_n$ are substituted back in (10) to get the unknown function $U(x)$ in terms of x.

Numerical Experiments

In this section, the algorithms discussed in section 4 above are implemented on some selected problems. The resulting systems of equations are solved by Mathematica 7.0.

Problem 1a

Use Boole's rule to approximate the Fredholm integral equation

$$U(x) = 1 + 7x + 20x^2 + x^3 - \int_0^1 (10xt^2 + 20x^2t)U(t)dt \tag{12}$$

Solution

Using (8) and (9) with the step size, $h = \frac{1}{4}$, (12) is written as

$$U_i + \frac{1}{90}[7(10t_it_0^2 + 20t_i^2t_0)U_0 + 32(10t_it_1^2 + 20t_i^2t_1)U_1 + 12(10t_it_2^2 + 20t_i^2t_2)U_2 + 32(10t_it_3^2 + 20t_i^2t_3)U_3$$
$$+7(10t_it_4^2 + 20t_i^2t_4)U_4] = 1 + 7t_i + 20t_i^2 + t_i^3 \tag{13}$$

Inserting $t_0 = 0, t_1 = \frac{1}{4}, t_2 = \frac{1}{2}, t_3 = \frac{3}{4}$ and $t_4 = 1$ in (13) yields

$$U_0 = 1$$
$$\frac{7}{6}U_1 + \frac{1}{6}U_2 + \frac{5}{6}U_3 + \frac{7}{24}U_4 = \frac{257}{64}$$
$$\frac{5}{9}U_1 + \frac{3}{2}U_2 + \frac{7}{3}U_3 + \frac{7}{9}U_4 = \frac{77}{8}$$
$$\frac{7}{6}U_1 + U_2 + \frac{11}{2}U_3 + \frac{35}{24}U_4 = \frac{1147}{64}$$
$$2U_1 + \frac{5}{3}U_2 + \frac{22}{3}U_3 + \frac{10}{3}U_4 = 29$$

Solving the above system of equations gives

$U_0 = 1, U_1 = \frac{69}{64}, \mathrm{U}_2 = \frac{11}{8}, U_3 = \frac{127}{64}$ and $U_4 = 3.$

Next is to substitute the values of $U_i, i = 0(1)4,$ in (4).This yields

$$U(x) = 1 + x^2 + x^3,$$

which tallies with the exact solution.

Problem 1b

Use Weddle's rule to approximate the Fredholm integral equation

$$U(x) = 1 + 7x + 20x^2 + x^3 - \int_0^1 (10xt^2 + 20x^2 t)U(t)dt \tag{14}$$

Solution

Using (10), (11) and step size $h = \frac{1}{6}$, we have (14) rewritten in the form

$$U_i + \frac{1}{20}[(10t_i t_0^2 + 20t_i^2 t_0)U_0 + 5(10t_i t_1^2 + 20t_i^2 t_1)U_1 + (10t_i t_2^2 + 20t_i^2 t_2)U_2 + 6(10t_i t_3^2 + 20t_i^2 t_3)U_3$$
$$+(10t_i t_4^2 + 20t_i^2 t_4)U_4 + 5(10t_i t_5^2 + 20t_i^2 t_5)U_5 + (10t_i t_6^2 + 20t_i^2 t_6)U_6] = 1 + 7t_i + 20t_i^2 + t_i^3 \tag{15}$$

Using the values $t_0 = 0, t_1 = \frac{1}{6}, t_2 = \frac{1}{3}, t_3 = \frac{1}{2}, t_4 = \frac{2}{3}, t_5 = \frac{5}{6}$ and $t_6 = 1$in (15), we have the following system of equations

$$U_0 = 1$$
$$\frac{49}{144}U_1 + \frac{1}{54}U_2 + \frac{5}{24}U_3 + \frac{1}{18}U_4 + \frac{175}{432}U_5 + \frac{1}{9}U_6 = \frac{589}{216}$$
$$\frac{25}{216}U_1 + \frac{19}{18}U_2 + \frac{7}{12}U_3 + \frac{4}{27}U_4 + \frac{25}{24}U_5 + \frac{5}{18}U_6 = \frac{151}{27}$$
$$\frac{35}{144}U_1 + \frac{1}{9}U_2 + \frac{17}{8}U_3 + \frac{5}{18}U_4 + \frac{275}{144}U_5 + \frac{1}{2}U_6 = \frac{77}{8}$$
$$\frac{5}{12}U_1 + \frac{5}{27}U_2 + \frac{11}{6}U_3 + \frac{13}{9}U_4 + \frac{325}{108}U_5 + \frac{7}{9}U_6 = \frac{401}{27}$$
$$\frac{275}{432}U_1 + \frac{5}{18}U_2 + \frac{65}{24}U_3 + \frac{35}{54}U_4 + \frac{769}{144}U_5 + \frac{10}{9}U_6 = \frac{4601}{216}$$
$$\frac{65}{72}U_1 + \frac{7}{18}U_2 + \frac{15}{4}U_3 + \frac{8}{9}U_4 + \frac{425}{72}U_5 + \frac{5}{2}U_6 = 29$$

Solving the system of equations above yields

$U_0 = 1, U_1 = \frac{223}{216}, U_2 = \frac{31}{27}, U_3 = \frac{11}{8}, U_4 = \frac{47}{27}, U_5 = \frac{491}{216}$ and $U_6 = 3.$

Substituting for the values of U_i and $t_i \left(i = 0(1)6\right)$ gives the unknown function as

$U(x) = 1 + x^2 + x^3.$

This again tallies with exact solution.

Problem 2a

Use Boole's rule to approximate the Fredholm integral equation

$$U(x) = 1 + 9x + 2x^2 + x^3 - \int_0^1 (20xt + 10x^2t^2)U(t)dt \quad (16)$$

Solution

The step size, h for this problem is $\frac{1}{4}$, when $n = 4.$ Now, using (8) and (9) in (16) yields

$$U_i + \frac{1}{90}[7(20t_it_0 + 10t_i^2t_0^2)U_0 + 32(20t_it_1 + 10t_i^2t_1^2)U_1 + 12(20t_it_2 + 10t_i^2t_2^2)U_2 + 32(20t_it_3 + 10t_i^2t_3^2)U_3$$
$$+7(20t_it_4 + 10t_i^2t_4^2)U_4] = 1 + 9t_i + 2t_i^2 + t_i^3 \quad (17)$$

With $t_0 = 0, t_1 = \frac{1}{4}, t_2 = \frac{1}{2}, t_3 = \frac{3}{4}$ and $t_4 = 1,$ (4) generates the following system of equations

$U_0 = 1$

$\frac{35}{24}U_1 + \frac{17}{48}U_2 + \frac{35}{24}U_3 + \frac{7}{16}U_4 = \frac{217}{64}$

$\frac{17}{18}U_1 + \frac{7}{4}U_2 + \frac{19}{6}U_3 + \frac{35}{36}U_4 = \frac{49}{8}$

$2U_1 + \frac{5}{3}U_2 + \frac{22}{3}U_3 + \frac{10}{3}U_4 = 13$

$\frac{35}{24}U_1 + \frac{19}{16}U_2 + \frac{49}{8}U_3 + \frac{77}{48}U_4 = \frac{595}{64}$

Solving the system of equations above gives

$U_0 = 1, U_1 = \frac{61}{64}, U_2 = \frac{7}{8}, U_3 = \frac{55}{64}$ and $U_4 = 1.$

When the values these constants are substituted in (8), the unknown function is obtained as $U(x) = 1 - x^2 + x^3.$

Thus, the approximate solution tallies with exact solution in this case too.

Problem 2b

Use Weddle's rule to approximate the Fredholm integral equation

$$U(x) = 1 + 9x + 2x^2 + x^3 - \int_0^1 (20xt + 10x^2t^2)U(t)dt \quad (18)$$

Solution

The given integral equation is approximated using (10) and (11) with step size $h = \frac{1}{6}$ to get

$$U_i + \frac{1}{20}[(20t_it_0 + 10t_i^2t_0^2)U_0 + 5(20t_it_1 + 10t_i^2t_1^2)U_1 + (20t_it_2 + 10t_i^2t_2^2)U_2 + 6(20t_it_3 + 10t_i^2t_3^2)U_3$$
$$+(20t_it_4 + 10t_i^2t_4^2)U_4 + 5(20t_it_5 + 10t_i^2t_5^2)U_5 + (20t_it_6 + 10t_i^2t_6^2)U_6] = 1 + 9t_i + 2t_i^2 + t_i^3 \quad (19)$$

with $t_0 = 0, t_1 = \frac{1}{6}, t_2 = \frac{1}{3}, t_3 = \frac{1}{2}, t_4 = \frac{2}{3}, t_5 = \frac{5}{6}$, and $t_6 = 1,$ from (19) we have the following set of equations

$$U_0 = 1$$
$$\frac{2957}{2592}U_1 + \frac{37}{648}U_2 + \frac{25}{48}U_3 + \frac{19}{162}U_4 + \frac{1925}{2592}U_5 + \frac{13}{72}U_6 = \frac{553}{216}$$
$$\frac{185}{648}U_1 + \frac{181}{162}U_2 + \frac{13}{12}U_3 + \frac{20}{81}U_4 + \frac{1025}{648}U_5 + \frac{7}{18}U_6 = \frac{115}{27}$$
$$\frac{125}{288}U_1 + \frac{13}{72}U_2 + \frac{43}{16}U_3 + \frac{7}{18}U_4 + \frac{725}{288}U_5 + \frac{5}{8}U_6 = \frac{49}{8}$$
$$\frac{95}{162}U_1 + \frac{20}{81}U_2 + \frac{7}{3}U_3 + \frac{125}{81}U_4 + \frac{575}{162}U_5 + \frac{8}{9}U_6 = \frac{221}{27}$$
$$\frac{1925}{2592}U_1 + \frac{205}{648}U_2 + \frac{145}{48}U_3 + \frac{115}{162}U_4 + \frac{14717}{2592}U_5 + \frac{85}{72}U_6 = \frac{2261}{216}$$
$$\frac{65}{72}U_1 + \frac{7}{18}U_2 + \frac{15}{4}U_3 + \frac{8}{9}U_4 + \frac{425}{72}U_5 + \frac{5}{2}U_6 = 13$$

Solving the above system of equations yields

$U_0 = 1, U_1 = \frac{211}{216}, U_2 = \frac{25}{27}, U_3 = \frac{7}{8}, U_4 = \frac{23}{27}, U_5 = \frac{191}{216},$ and $U_6 = 1.$

The values of these constants are substituted in (8) to arrive at result for the unknown function as $U(x) = 1 - x^2 + x^3,$

which is the same as the as the one obtained in Problem 5.2a, hence the exact solution.

Problem 3a

Use Boole's rule to approximate the Fredholm integral equation

$$U(x) = 5x - 2x^2 + \int_{-1}^{1}(x^2t^3 - x^3t^2)U(t)dt \quad (20)$$

Solution

(20) is approximated by Boole's rule as stated in (8) and (9) using step size, $h = \frac{1}{2}$ to get

$$U_i - \frac{1}{45}[7(t_i^2t_0^3 - t_i^3t_0^2)U_0 + 32(t_i^2t_1^3 - t_i^3t_1^2)U_1 + 12(t_i^2t_2^3 - t_i^3t_2^2)U_2 + 32(t_i^2t_3^3 - t_i^3t_3^2)U_3$$
$$+7(t_i^2t_4^3 - t_i^3t_4^2)U_4] = 5t_i - 2t_i^2 \quad (21)$$

Using $t_0 = -1, t_1 = -\frac{1}{2}, t_2 = 0, t_3 = \frac{1}{2},$ and $t_4 = 1,$ we have the following four equations in four unknowns

$$U_0 - \frac{4}{45}U_1 - \frac{4}{15}U_3 - \frac{14}{45}U_4 = -7$$

$$\frac{7}{360}U_0 + U_1 - \frac{2}{45}U_3 - \frac{7}{120}U_4 = -3$$

$$U_2 = 0$$

$$\frac{7}{120}U_0 + \frac{2}{45}U_1 + U_3 - \frac{7}{360}U_4 = 2$$

$$\frac{14}{45}U_0 + \frac{4}{15}U_1 + \frac{4}{45}U_3 + U_4 = 3$$

Solving the above system of equations gives

$U_0 = -5, U_1 = -\frac{5}{2}, U_2 = 0, U_3 = \frac{5}{2},$ and $U_4 = 5.$

Substituting the values of these constants in (8) gives the unknown function as

$U(x) = 5x,$

which is the exact solution.

Problem 3b

Use Weddle's rule to find the approximate solution of the Fredholm integral equation

$$U(x) = 5x - 2x^2 + \int_{-1}^{1}(x^2t^3 - x^3t^2)U(t)dt \tag{22}$$

Solution

Applying (10) and (11) with step size, $h = \frac{1}{3}$, in (22), we have

$$U_i - \frac{1}{10}[(t_i^2t_0^3 - t_i^3t_0^2)U_0 + 5(t_i^2t_1^3 - t_i^3t_1^2)U_1 + (t_i^2t_2^3 - t_i^3t_2^2)U_2 + 6(t_i^2t_3^3 - t_i^3t_3^2)U_3 + (t_i^2t_4^3 - t_i^3t_4^2)U_4$$
$$+5(t_i^2t_5^3 - t_i^3t_5^2)U_5 + (t_i^2t_6^3 - t_i^3t_6^2)U_6] = 5t_i - 2t_i^2 \tag{23}$$

Using the values $t_0 = -1, t_1 = -\frac{2}{3}, t_2 = -\frac{1}{3}, t_3 = 0, t_4 = \frac{1}{3}, t_5 = \frac{2}{3},$ and $t_6 = 1$ in (23) to get

$$U_0 - \frac{2}{27}U_1 - \frac{1}{135}U_2 - \frac{2}{135}U_4 - \frac{10}{27}U_5 - \frac{1}{5}U_6 = -7$$

$$\frac{2}{135}U_0 + U_1 - \frac{2}{1215}U_2 - \frac{2}{405}U_4 - \frac{32}{243}U_5 - \frac{2}{27}U_6 = -\frac{38}{9}$$

$$\frac{1}{135}U_0 + \frac{2}{243}U_1 + U_2 - \frac{1}{1215}U_4 - \frac{2}{81}U_5 - \frac{2}{135}U_6 = -\frac{17}{9}$$

$$\frac{2}{135}U_0 + \frac{2}{81}U_1 + \frac{1}{1215}U_2 + U_4 - \frac{2}{243}U_5 - \frac{1}{135}U_6 = \frac{13}{9}$$

$$\frac{2}{27}U_0 + \frac{32}{243}U_1 + \frac{2}{405}U_2 + \frac{2}{1215}U_4 + U_5 - \frac{2}{135}U_6 = \frac{22}{9}$$

$$\frac{1}{5}U_0 + \frac{10}{27}U_1 + \frac{2}{135}U_2 + \frac{1}{135}U_4 + \frac{2}{27}U_5 + U_6 = 3$$

Solving the above system of equations for the unknowns yields

$U_0 = -5,\ U_1 = -\frac{10}{3},\ U_2 = -\frac{5}{3},\ U_3 = 0,\ U_4 = \frac{5}{3},\ U_5 = \frac{10}{3}$ and $U_6 = 5.$

Substituting the values of these constants in (4) gives the unknown function, $U(x)$ as
$U(x) = 5x.$ This result agrees with the earlier one in Problem 5.3a and are both the same as the exact solution.

Problem 4a
Solve the Fredholm integral equation below using Boole's rule

$$U(x) = -15 + 10x^3 + x^4 - \int_0^1 (20x^3 - 56t^3)U(t)dt \tag{24}$$

Solution
The given integral equation can be rewritten as

$$U(x) + \int_0^1 (20x^3 - 56t^3)U(t)dt = -15 + 10x^3 + x^4 \tag{25}$$

Using (9), (25) now becomes

$$U_i + \frac{1}{90}[7(20t_i^3 - 56t_0^3)U_0 + 32(20t_i^3 - 56t_1^3)U_1 + 12(20t_i^3 - 56t_2^3)U_2 + 32(20t_i^3 - 56t_3^3)U_3 +$$
$$7(20t_i^3 - 56t_4^3)U_4] = -15 + 10t_i^3 + t_i^4, \quad i = 0(1)4 \tag{26}$$

with $t_0 = 0, t_1 = \frac{1}{4}, t_2 = \frac{1}{2}, t_3 = \frac{3}{4}$, and $t_4 = 1$ in (8) and (9), we have the following set of equations

$$U_0 - \frac{14}{15}U_1 - \frac{14}{15}U_2 - \frac{42}{5}U_3 - \frac{196}{45}U_4 = -15$$
$$\frac{7}{288}U_0 + \frac{4}{5}U_1 - \frac{107}{120}U_2 - \frac{373}{45}U_3 - \frac{693}{160}U_4 = -\frac{3799}{256}$$
$$\frac{7}{36}U_0 + \frac{26}{45}U_1 + \frac{2}{5}U_2 - \frac{338}{45}U_3 - \frac{794}{180}U_4 = -\frac{219}{16}$$
$$\frac{21}{32}U_0 - \frac{121}{45}U_1 + \frac{23}{120}U_2 - \frac{22}{5}U_3 - \frac{5327}{1440}U_4 = -\frac{2679}{256}$$
$$\frac{14}{9}U_0 + \frac{34}{5}U_1 + \frac{26}{15}U_2 - \frac{58}{45}U_3 - \frac{9}{5}U_4 = -4$$

Solving system of five equations in five unknowns above, we obtained
$U_0 = \frac{1085935}{55976896}, U_1 = \frac{2666115}{111953792}, U_2 = \frac{10618399}{55976896}, U_3 = \frac{41131777}{55976896}$ and $U_4 = \frac{111140301}{55976896}$.
Using these values in (8) gives

$$U(x) = \frac{256477}{55976896} + \frac{3431683}{3498556}x^3 + x^4,$$

which can be written equivalently as

$$U(x) = 0.00458184 + 0.980886x^3 + x^4,$$

which is a good approximation to the exact solution $U(x) = x^3 + x^4$.

Problem 5.4b
Solve the Fredholm integral equation below using Weddle's rule

$$U(x) = -15 + 10x^3 + x^4 - \int_0^1 (20x^3 - 56t^3)U(t)dt \tag{27}$$

Solution
Using (10) and (11), the given integral equation becomes

$$U_i+\frac{1}{20}[(20t_i^3-56t_0^3)U_0+5(20t_i^3-56t_1^3)U_1+(20t_i^3-56t_2^3)U_2+6(20t_i^3-56t_3^3)U_3+(20t_i^3-56t_4^3)U_4$$
$$+5(20t_i^3-56t_5^3)U_5+(20t_i^3-56t_6^3)U_6]=-15+10t_i^3+t_i^4 \qquad (28)$$

When the values of $t_i,(i=0(1)6)$, given as $t_0=0, t_1=\frac{1}{6}, t_1=\frac{1}{3}, t_3=\frac{1}{2}, t_4=\frac{2}{3}, t_5=\frac{5}{6}, t_6=1$, (14) generates the following system of equations

$$U_0-\frac{7}{108}U_1-\frac{14}{135}U_2-\frac{21}{10}U_3-\frac{112}{135}U_4-\frac{875}{108}U_5-\frac{14}{5}U_6=-15$$
$$\frac{1}{216}U_0+\frac{23}{24}U_1-\frac{107}{1080}U_2-\frac{373}{180}U_3-\frac{33}{40}U_4-\frac{1745}{216}U_5-\frac{3019}{1080}U_6=-\frac{19379}{1296}$$
$$\frac{1}{27}U_0+\frac{13}{108}U_1+\frac{14}{15}U_2-\frac{169}{90}U_3-\frac{107}{135}U_4-\frac{95}{12}U_5-\frac{373}{185}U_6=-\frac{1184}{81}$$
$$\frac{1}{8}U_0+\frac{121}{216}U_1+\frac{23}{1080}U_2-\frac{7}{20}U_3-\frac{761}{1080}U_4-\frac{1615}{216}U_5-\frac{107}{40}U_6=-\frac{219}{16}$$
$$\frac{8}{27}U_0+\frac{17}{12}U_1+\frac{26}{135}U_2-\frac{29}{90}U_3+\frac{7}{15}U_4-\frac{715}{108}U_5-\frac{338}{135}U_6=-\frac{959}{81}$$
$$\frac{125}{216}U_0+\frac{611}{216}U_1+\frac{19}{40}U_2+\frac{247}{180}U_3-\frac{271}{1080}U_4-\frac{101}{24}U_5-\frac{2399}{1080}U_6=-\frac{11315}{1296}$$
$$U_0+\frac{533}{108}U_1+\frac{121}{135}U_2+\frac{39}{10}U_3+\frac{23}{135}U_4-\frac{335}{108}U_5-\frac{4}{5}U_6=-4$$

The system of equations above is solved to obtain the unknown function $U(x)$ as $U(x)=0.0674276+0.671012x^3+x^4$,
which though not the same as the exact solution, but a very good approximation to it.

Discussion of Results

The numerical problems presented above show that the work presented in this paper is very accurate and can be used for any linear Fredholm integral equation of the second kind. The first three set of problem give the exact solution, and the last set that does not give exact solution actually gives results that are useable in numerical computations. The results of the numerical examples emphasizes the beauty of the method presented in this paper, as it overcomes the problems encountered in some of the semi – analytic methods such as Adomian decomposition method (ADM), Modified Adomian decomposition method (mADM), Variational Iteration Method (VIM), etc. For instance, the problem of determining which terms constitute the U_0 in ADM and $U_0\, and\, U_1$ in mADM.

Conclussion

Numerical solution of linear nonhomogeneous Fredholm integral equation has been presented. The method proposed in this work handles problems in this class with ease and does not require large computer memory for its implementation. In addition, it gives the exact solution in most cases, and where such is not obtained, the approximate solution derive is very reliable.

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