

BLOCK IMPLICIT HYBRID LINEAR MULTISTEP METHODS FOR SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The idea of multistep collocation is employed to reformulate the Hybrid Backward Differentiation Formula (HBDF) for cases  $k = 4$  and  $6$  into continuous forms which were evaluated at some interpolation and collocation points. This gives rise to discrete schemes that are combined to form the efficient block method for the solution of Ordinary Differential Equations. The requirement of starting values and the overlap of solution model which are associated with conventional Linear Multistep Methods are eliminated by this approach. A convergence analysis of the derived hybrid schemes is presented to establish their effectiveness and reliability. Numerical examples carried out further substantiates their convergence.

Keywords: Linear Multistep method (LMM), Backward Differentiation Formula (BDF), Block Solution, Implicit, Hybrid

Introduction

Consider the Initial Value Problem of the form

$$y' = f(x, y) \quad y(x_0) = y_0 \quad (1)$$

Where the solution  $y$  is assumed to be a differentiable function on an interval  $[x_0, b]$ ,  $b < \infty$ . Many methods for solving (1) exist, one particular method is the Linear Multistep Method. The Method require less evaluation of the derivative function  $f$  than one step methods in the range of integral  $[x_0, b]$ . For this reasons they have been very popular and important for solving (1) numerically. But these methods have certain limitations such as the overlap of solution models and the requirements of a starting value. Other limitations include they yield the discretely solution values  $y_1, \dots, y_N$  hence uneconomical for producing dense output. A continuous formulation is desirable in this respect. The collocation method is probably the most important numerical procedure for the construction of continuous methods ( Lie and Norsett, 1989; Awoyemi, Jator and sirisena, 1994; Onumanyi, sirisena and jator, 1999).

In this research paper, an attempt is made to reformulate the hybrid form of some BDF schemes, specifically ( $k = 4$  and  $6$ ) into continuous forms by the idea of multistep collocation.

The Continous Multistep Collocation (CMM) Method

Lambert (1973; 1991) adopted the continous finite difference (CFD) approximation approach by the idea of interpolation and collocation. Later, Lie and Norsett (1989), Onumanyi (1994; 1999) referred to it as Multistep Collocation (MC). The method is presented below

$$\underline{a} = (a_0, a_1, \dots, a_{t+m-1})^T, \varphi(x) = (\varphi_0(x), \varphi_1(x), \dots, \varphi_{t+m-1}(x))^T \quad (2)$$

where  $a_r, r = 0, \dots, t + m - 1$  are undetermined constants,  $\varphi_r(x)$  are specified basis functions,  $T$  denotes transpose of,  $t$  denotes the number of interpolation points and  $m$  denotes the

number of distinct collocation points. We consider a continuous approximation (interpolant)  $Y(x)$  to  $y(x)$  in the form

$$y(x) = \sum_{r=0}^{t+m-1} a_r \varphi_r(x) = \underline{a}^T \varphi(x) \tag{3}$$

which is valid in the sub-intervals  $x_n \leq x \leq x_{n+k}$ , where  $n = 0, k, \dots, N - k$ . The quantities

$x_0 = a, x_N = b, k, m, n, t$  and  $\varphi_r(x), r = 0, 1, \dots, t + m - 1$

are specified values. The constant co-efficient  $a_r$  of (3) can be determined using the conditions

$$y(x_{n+j}) = y_{n+j}, j = 0, 1, \dots, t - 1 \tag{4}$$

$$y'(\bar{x}_j) = f_{n+j} \quad j = 0, 1, \dots, m - 1 \tag{5}$$

Where

$$f_{n+j} = f(x_{n+j}, y_{n+j}) \tag{6}$$

The distinct collocation points  $x_0, \dots, x_{m-1}$ , can be chosen freely from the set  $[x_n, x_{n+k}]$ .

Equation (4), (5) and (6) are denoted by a single set of algebraic equations of the form

$$D\underline{a} = \underline{F} \tag{7}$$

$$\text{Where } \underline{F} = (y_n, y_{n+1}, \dots, y_{n+t-1}, f_n, f_{n+1}, f_{n+m-1})^T \tag{8}$$

$$\underline{a} = D^{-1}\underline{F} \tag{9}$$

where  $D$  is the non-singular matrix of dimension  $(t + m)$  below

$$D = \begin{pmatrix} \varphi_0(x_n) & \dots & \varphi_{t+m-1}(x_n) \\ \vdots & \ddots & \vdots \\ \varphi_0(x_{n+t-1}) & \dots & \varphi_{t+m-1}(x_{n+t-1}) \\ \vdots & \ddots & \vdots \\ \varphi'_0(\bar{x}_0) & \dots & \varphi'_{t+m-1}(\bar{x}_0) \\ \vdots & \ddots & \vdots \\ \varphi'_0(\bar{x}_{m-1}) & \dots & \varphi'_{t+m-1}(\bar{x}_{m-1}) \end{pmatrix} \tag{10}$$

By substituting (9) into (3), we obtain the MC formula

$$y(x) = F^T C^T \varphi(x), \quad x_n \leq x \leq x_{n+k} \quad n = 0, k, \dots, N - k \tag{11}$$

where

$$C \equiv D^{-1} = (c_{ij}), i, j = 1, \dots, t + m - 1$$

$$C = \begin{pmatrix} c_{11} & \dots & c_{1t} & c_{1,t+1} & \dots & c_{1,t+m} \\ c_{21} & \dots & c_{2t} & c_{2,t+1} & \dots & c_{2,t+m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{t+m,1} & \dots & c_{t+m,t} & c_{t+m,t+1} & \dots & c_{t+m,t+m} \end{pmatrix}$$

with the numerical elements denoted by  $c_{ij}, i, j = 1, \dots, k + m$ . By expanding  $C^T \varphi(x)$  in (11)

yields the following

$$y(x) = (F)^T \begin{pmatrix} \sum_{r=0}^{t+m-1} c_{r+1,1} \varphi_r(x) \\ \vdots \\ \sum_{r=0}^{t+m-1} c_{r+1,k+m} \varphi_r(x) \end{pmatrix} \tag{12}$$

$$y(x) = \sum_{j=0}^{t-1} (\sum_{r=0}^{t+m-1} c_{r+1,j+1} \varphi_r(x)) + \sum_{j=0}^{m-1} h \left( \sum_{r=0}^{k+m-1} \frac{c_{r+1,j+1}}{h} \varphi_r(x) \right) f_{n+j} \tag{13}$$

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f_{n+j} \tag{14}$$

where we construct  $\alpha_j(x)$  and  $\beta_j(x)$  explicitly by

$$\alpha_j(x) = \sum_{r=0}^{t+m-1} C_{r+1,j+1} \varphi_r(x) \quad j = 0, 1, \dots, t-1 \quad (15)$$

$$\beta_j(x) = \sum_{r=0}^{k+m-1} \left( \frac{C_{r+1,j+1}}{h} \right) \varphi_r(x), \quad j = 0, 1, \dots, m-1 \quad (16)$$

$a_r$  can be determined as follows:

$$y(x) = \left\{ \sum_{r=0}^{t-1} \alpha_{j,r+1} y_{n+j} + h \sum_{j=0}^{m-1} \beta_{j,r+1} f_{n+j} \right\} \varphi_r(x) \quad (17)$$

For  $K = 4$ , the general form of the method upon addition of one off grid point is expressed as;

$$\bar{y}(x) = \alpha_1(x)y_n + \alpha_2(x)y_{n+1} + \alpha_3(x)y_{n+2} + \alpha_4(x)y_{n+3} + \alpha_5(x)y_{n+\frac{1}{2}} + h\beta_0(x)f_{n+4} \quad (18)$$

The matrix D of the proposed method is expressed as:

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_n+h & (x_n+h)^2 & (x_n+h)^3 & (x_n+h)^4 & (x_n+h)^5 \\ 1 & x_n+2h & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 \\ 1 & x_n+3h & (x_n+3h)^2 & (x_n+3h)^3 & (x_n+3h)^4 & (x_n+3h)^5 \\ 1 & x_n+\frac{1}{2}h & (x_n+\frac{1}{2}h)^2 & (x_n+\frac{1}{2}h)^3 & (x_n+\frac{1}{2}h)^4 & (x_n+\frac{1}{2}h)^5 \\ 0 & 1 & 2x_n+8h & 3(x_n+4h)^2 & 4(x_n+4h)^3 & 5(x_n+4h)^4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+1/2} \\ f_{n+4} \end{bmatrix} \quad (19)$$

i.e  $Da = F$

The matrix D in equation (19) which when solved by matrix inversion technique or Gaussian Elimination method will yield the continuous coefficients substituted in (18) to obtain continuous form of the four step block hybrid BDF with one off step interpolation point.

$$\begin{aligned}
 & \left( \frac{1}{2388} \frac{1}{h^5} (2388h^5 + 9688h^4 + 13191x_n^2h^3 + 7666x_n^3h^2 + 1953x_n^4h + 178x_n^5) - \right. \\
 & \frac{1}{1194} \frac{1}{h^5} (4844h^4 + 13191x_nh^3 + 11499x_n^2h^2 + 3906x_n^3h + 445x_n^4)x + \\
 & \frac{1}{2388} \frac{1}{h^5} (13191h^3 + 22998x_nh^2 + 11718x_n^2h + 1780x_n^3)x^2 - \frac{1}{1194} \frac{1}{h^5} (3833h^2 + 3906x_nh + 890x_n^2)x^3 + \frac{1}{2388} \frac{1}{h^5} (1953h + 890x_n)x^4 - \\
 & \frac{89}{1194h^5} x^5) y_n + \left( \frac{1}{398} \frac{1}{h^5} (x_n(1534h^4 + 4694x_nh^3 + 3785x_n^2h^2 + 1139x_n^3h + 114x_n^4)) - \right. \\
 & \frac{1}{398} \frac{1}{h^5} (x_n(1536h^4 + 4694x_nh^3 + 3785x_n^2h^2 + 1139x_n^3h + 114x_n^4)) x + \\
 & \frac{1}{398} \frac{1}{h^5} (4694h^3 + 11358x_nh^2 + 6834x_n^2h + 1140x_n^3)x^2 - \frac{1}{398} \frac{1}{h^5} (3785h^2 + 4556x_nh + 1140x_n^2)x^3 + \frac{1}{398} \frac{1}{h^5} (1139h + 570x_n)x^4 - \\
 & \frac{57}{199h^5} x^5) y_{n+1} + \left( -\frac{1}{2388} \frac{1}{h^5} (x_n(7591x_nh^3 + 7978x_n^2h^2 + 2837x_n^3h + 314x_n^4 + 2136h^4)) + \right. \\
 & \frac{1}{1194} \frac{1}{h^5} (7591x_nh^3 + 11967x_n^2h^2 + 5674x_n^3h + 785x_n^4 + 1068h^4)x - \\
 & \frac{1}{2388} \frac{1}{h^5} (7591h^3 + 23934x_nh^2 + 17022x_n^2h + 3140x_n^3)x^2 + \frac{1}{1194} \frac{1}{h^5} (3989h^2 + 5674x_nh + 1570x_n^2)x^3 - \\
 & \frac{157}{2388} \frac{1}{h^5} (2837h + 1570x_n)x^4 + \frac{157}{1194h^5} x^5) y_{n+2} + \left( \frac{1}{5970} \frac{1}{h^5} (x_n(4613x_n^2h^2 + 1893x_n^3h + 230x_n^4 + \right. \\
 & 4038x_nh^3 + 1088h^4)) - \frac{1}{5970} \frac{1}{h^5} (13839x_n^2h^2 + 7572x_n^3h + 1150x_n^4 + 785x_n^4 + 8076x_nh^3 + \\
 & 1088h^4)x + \frac{1}{5970} \frac{1}{h^5} (13839x_nh^2 + 11358x_n^2h + 2300x_n^3h + 4038h^4)x^2 - \frac{1}{5970} \frac{1}{h^5} (4613h^2 + 7572x_nh + 2300x_n^2)x^3 + \\
 & \frac{1}{5970} \frac{1}{h^5} (1893h + 1150x_n)x^4 + -\frac{23}{5970h^5} x^5) y_{n+3} + \left( -\frac{32}{2985} \frac{1}{h^5} (x_n(25x_n^4 + 262x_n^3h + 947x_n^2h^2 + \right. \\
 & 1382x_nh^3 + 672h^4)) + \frac{32}{2985} \frac{1}{h^5} (125x_n^4 + 1048x_n^3h + 2841x_n^2h^2 + 2764x_nh^3 + 672h^4)x - \\
 & \frac{32}{2985} \frac{1}{h^5} (250x_n^3h + 1572x_n^2h + 2841x_nh^2 + 1382h^3)x^2 + \frac{32}{2985} \frac{1}{h^5} (250x_n^2h + 1048x_nh + 947h^2)x^3 - \frac{32}{2985} \frac{1}{h^5} (125x_n + 262h)x^4 + \\
 & \frac{160}{5970h^5} x^5) y_{n+1/2} + \\
 & \left. \left( -\frac{1}{398} \frac{1}{h^4} (x_n(2x_n^4 + 13x_n^3h + 28x_n^2h^2 + 23x_nh^3 + 6h^4)) + \frac{1}{199} \frac{1}{h^4} (5x_n^4 + 26x_n^3h + 42x_n^2h^2 + 23x_nh^3 + 3h^4)x - \right. \right. \\
 & \left. \left. \frac{1}{398} \frac{1}{h^4} (20x_n^3h + 78x_n^2h + 84x_nh^2 + 23h^3)x^2 + \frac{2}{199} \frac{1}{h^4} (5x_n^2h + 13x_nh + 7h^2)x^3 - \frac{1}{398} \frac{1}{h^4} (10x_n + 13h)x^4 + \frac{1}{199h^4} x^5) f_{n+4} \right\} \quad (20)
 \end{aligned}$$

Evaluating (20) at points  $x = x_{n+4}$  and its derivative at  $x = x_{n+3}, x = x_{n+1}, x = x_{n+2}, x = x_{n+1/2}$  yields the following five discrete hybrid schemes which are used as a block integrator;

$$\begin{aligned}
 & -\frac{784}{199} y_{n+1} + \frac{588}{199} y_{n+2} - \frac{2352}{995} y_{n+3} + y_{n+4} + \frac{3072}{995} y_{n+1/2} = \frac{147}{199} y_n + \frac{84}{199} h f_{n+4} \\
 & -y_{n+1} - \frac{925}{2415} y_{n+2} + \frac{143}{2415} y_{n+3} + \frac{3712}{2415} y_{n+1/2} = \frac{515}{2415} y_n - \frac{1990}{2415} h f_{n+1} + \frac{10}{2415} h f_{n+4} \\
 & \frac{6390}{2305} y_{n+1} - y_{n+2} - \frac{942}{2305} y_{n+3} - \frac{3968}{2305} y_{n+1/2} = -\frac{825}{2305} y_n - \frac{2985}{2305} h f_{n+2} - \frac{45}{2305} h f_{n+4} \\
 & -\frac{19125}{9883} y_{n+1} + \frac{18075}{9883} y_{n+2} - y_{n+3} + \frac{14208}{9883} y_{n+1/2} = \frac{3275}{9883} y_n - \frac{5970}{9883} h f_{n+3} + \frac{450}{9883} h f_{n+4} \\
 & -\frac{125888}{450} y_{n+1} + \frac{125888}{125888} y_{n+2} - \frac{5888}{125888} y_{n+3} + y_{n+1/2} = -\frac{53025}{125888} y_n - \frac{95520}{125888} h f_{n+1/2} - \\
 & \frac{450}{125888} h f_{n+4} \quad (21)
 \end{aligned}$$

Equation (21) constitute the members of a zero-stable block integrators of order

$(5,5,5,5,5)^T$  with  $C_6 = \left[ -\frac{49}{995}, -\frac{227}{47760}, \frac{241}{23880}, -\frac{283}{9552}, \frac{1561}{305664} \right]^T$  as the error constants respectively. To start the integration

process with  $n=0$ , we use (21) and this produces  $y_1, y_{1/2}, y_2, y_3$  and  $y_4$  simultaneously without the need of any starting method (predictor).

For  $K = 6$ , the general form of the method upon addition of one off grid point is expressed as;

$$\bar{y}(x) = \alpha_1(x)y_n + \alpha_2(x)y_{n+1} + \alpha_3(x)y_{n+2} + \alpha_4(x)y_{n+3} + \alpha_5(x)y_{n+4} + \alpha_6(x)y_{n+5} + \alpha_7(x)y_{n+\frac{1}{2}} + h\beta_0(x)f_{n+6}$$

(22)

Recall from (7),  $Da = F$

The matrix D of the proposed method is expressed as:

D

$$= \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 & (x_n + h)^5 & (x_n + h)^6 & (x_n + h)^7 \\ 1 & x_n + 2h & (x_n + 2h)^2 & (x_n + 2h)^3 & (x_n + 2h)^4 & (x_n + 2h)^5 & (x_n + 2h)^6 & (x_n + 2h)^7 \\ 1 & x_n + 3h & (x_n + 3h)^2 & (x_n + 3h)^3 & (x_n + 3h)^4 & (x_n + 3h)^5 & (x_n + 3h)^6 & (x_n + 3h)^7 \\ 1 & x_n + 4h & (x_n + 4h)^2 & (x_n + 4h)^3 & (x_n + 4h)^4 & (x_n + 4h)^5 & (x_n + 4h)^6 & (x_n + 4h)^7 \\ 1 & x_n + 5h & (x_n + 5h)^2 & (x_n + 5h)^3 & (x_n + 5h)^4 & (x_n + 5h)^5 & (x_n + 5h)^6 & (x_n + 5h)^7 \\ 1 & x_n + \frac{1}{2}h & (x_n + \frac{1}{2}h)^2 & (x_n + \frac{1}{2}h)^3 & (x_n + \frac{1}{2}h)^4 & (x_n + \frac{1}{2}h)^5 & (x_n + \frac{1}{2}h)^6 & (x_n + \frac{1}{2}h)^7 \\ 0 & 1 & 2x_n + 12h & 3(x_n + 6h)^2 & 4(x_n + 6h)^3 & 5(x_n + 6h)^4 & 6(x_n + 6h)^5 & 7(x_n + 6h)^6 \end{bmatrix} \quad (23)$$

The matrix D in equation (23) which when solved by matrix inversion technique or Gaussian Elimination method will yield the continuous coefficients substituted in (22) to obtain continuous form of the six step block hybrid BDF with one off step interpolation point.

$$\bar{y}(x) = \frac{1}{h^7} \left( \frac{1}{1250640} P y_n + \frac{1}{13896} Q y_{n+1} + \frac{1}{20844} R y_{n+2} \right) + \frac{1}{h^7} \left( \frac{1}{312660} S y_{n+3} + \frac{1}{194844} T y_{n+4} + \frac{1}{628320} U y_{n+5} \right) + \frac{1}{h^7} \left( \frac{1}{547155} V y_{n+1/2} + \frac{1}{20844} W f_{n+6} \right) \quad (24)$$

where

$$P = (1250640h^7 + 3254x_n^7 + 71281x_n^6h + 624077x_n^5h^2 + 2798515x_n^4h^3 + 68333441x_n^3h^4 + 8892484x_n^2h^5 + 5552148x_nh^6) + (427686x_n^5h + 3120385x_n^4h^2 + 1194060x_n^3h^3 + 20500323x_n^2h^4 + 17784968x_nh^5 + 5552148h^6 + 22778x_n^6)x + (8892484h^5 + 1069215x_n^4h + 6240770x_n^3h^2 + 16791090x_n^2h^3 + 20500323x_nh^4 + 68334x_n^5)x^2 - (6833441h^4 + 11194060x_nh^3 + 6240770x_n^2h^2 + 1425620x_n^3h + 113890x_n^4)x^3 + 5(559703h^3 + 624077x_nh^2 + 213843x_n^2h + 22778x_n^3)x^4 - (624077h^2 + 427686x_nh + 68334x_n^2)x^5 + (10183h + 3254x_n)x^6 - (3254)x^7$$

$$Q = (4475x_n^6h + 36586x_n^5h^2 + 147569x_n^4h^3 + 302152x_n^3h^4 + 283124x_n^2h^5 + 82320x_nh^6 + 214x_n^7) - 2(13425x_n^5h + 91465x_n^4h^2 + 295138x_n^3h^3 + 453228x_n^2h^4 + 283124x_nh^5 + 41160h^6 + 749x_n^6)x + ((283124h^5 + 67125x_n^4h + 365860x_n^3h^2 + 885414x_n^2h^3 + 906456x_nh^4 + 4494x_n^5))x^2 - 2(151076h^4 + 295138x_nh^3 + 182930x_n^2h^2 + 44750x_n^3h + 3745x_n^4)x^3 + (147569h^3 + 182930x_nh^2 + 67125x_n^2h + 7490x_n^3)x^4 - 2(18293h^2 + 13425x_nh + 2247x_n^2)x^5 + (4475h + 1498x_n)x^6 - (214)x^7$$

$$R = -(5219x_n^6h + 39868x_n^5h^2 + 145934x_n^4h^3 + 259330x_n^3h^4 + 198767x_n^2h^5 + 50460x_nh^6 + 262x_n^7) + 2(15657x_n^5h + 99670x_n^4h^2 + 291868x_n^3h^3 + 388995x_n^2h^4 + 198767x_nh^5 + 25230h^6 + 917x_n^6)x - ((198767h^5 + 78285x_n^4h + 398680x_n^3h^2 + 875604x_n^2h^3 + 777990x_nh^4 + 5502x_n^5))x^2 + 2(129665h^4 + 291868x_nh^3 + 199340x_n^2h^2 + 52190x_n^3h + 4585x_n^4)x^3$$

$$+(145934h^3 + 199340x_n h^2 + 78285x_n^2 h + 9170x_n^3)x^4 + 2(19934h^2 + 15657x_n h + 2751x_n^2)x^5 - (5219h + 1834x_n)x^6 + (262)x^7$$

$$S = (57449x_n^6 h + 410920x_n^5 h^2 + 1384805x_n^4 h^3 + 2240896x_n^3 h^4 + 1603076x_n^2 h^5 + 390480x_n h^6 + 3034x_n^7) - 2(172347x_n^5 h + 1027300x_n^4 h^2 + 2769610x_n^3 h^3 + 3361344x_n^2 h^4 + 1603076x_n h^5 + 195240h^6 + 10619x_n^6)x$$

$$+ \left( (1603076h^5 + 861735x_n^4 h + 4109200x_n^3 h^2 + 8308830x_n^2 h^3 + 6722688x_n h^4 + 63714x_n^5) \right) x^2$$

$$- 2(1120448h^4 + 2769610x_n h^3 + 2954600x_n^2 h^2 + 574490x_n^3 h + 53095x_n^4)x^3$$

$$+ 5(276961h^3 + 410920x_n h^2 + 172347x_n^2 h + 21238x_n^3)x^4$$

$$- 2(205460h^2 + 172347x_n h + 31857x_n^2)x^5 + 7(8207h + 3034x_n)x^6 - (3034)x^7$$

$$T = -(16855x_n^6 h + 113399x_n^5 h^2 + 358609x_n^4 h^3 + 551291x_n^3 h^4 + 381184x_n^2 h^5 + 91020x_n h^6 + 938x_n^7) + (101130x_n^5 h + 566995x_n^4 h^2 + 1434436x_n^3 h^3 + 1653873x_n^2 h^4 + 762368x_n h^5 + 91020h^6 + 6566x_n^6)x$$

$$- \left( (381184h^5 + 252825x_n^4 h + 1133990x_n^3 h^2 + 2151654x_n^2 h^3 + 1653873x_n h^4 + 19698x_n^5) \right) x^2$$

$$+ (551291h^4 + 1434436x_n h^3 + 1133990x_n^2 h^2 + 337100x_n^3 h + 32830x_n^4)x^3$$

$$- (358609h^3 + 566995x_n h^2 + 252825x_n^2 h + 32830x_n^3)x^4$$

$$+ (113399h^2 + 101130x_n h + 19698x_n^2)x^5 - (16855h + 6566x_n)x^6 + (938)x^7$$

$$U = (12287x_n^6 h + 78574x_n^5 h^2 + 238385x_n^4 h^3 + 355672x_n^3 h^4 + 241268x_n^2 h^5 + 56976x_n h^6 + 718x_n^7) - 2(36861x_n^5 h + 196435x_n^4 h^2 + 476770x_n^3 h^3 + 533508x_n^2 h^4 + 241268x_n h^5 + 28488h^6 + 2513x_n^6)x$$

$$+ \left( (241268h^5 + 184305x_n^4 h + 785740x_n^3 h^2 + 1430310x_n^2 h^3 + 1067016x_n h^4 + 15078x_n^5) \right) x^2$$

$$- 2(177836h^4 + 476770x_n h^3 + 392870x_n^2 h^2 + 122870x_n^3 h + 12565x_n^4)x^3$$

$$+ 5(47677h^3 + 78574x_n h^2 + 36861x_n^2 h + 5026x_n^3)x^4 - 2(39287h^2 + 36861x_n h + 7539x_n^2)x^5$$

$$+ (12287h + 5026x_n)x^6 - (718)x^7$$

$$V = -(1049x_n^6 h + 8875x_n^5 h^2 + 37715x_n^4 h^3 + 84076x_n^3 h^4 + 91916x_n^2 h^5 + 37680x_n h^6 + 49x_n^7) + (6294x_n^5 h + 44375x_n^4 h^2 + 150860x_n^3 h^3 + 252228x_n^2 h^4 + 183832x_n h^5 + 37680h^6 + 343x_n^6)x$$

$$- \left( (91916h^5 + 15735x_n^4 h + 88750x_n^3 h^2 + 226290x_n^2 h^3 + 252228x_n h^4 + 1029x_n^5) \right) x^2$$

$$+ (84076h^4 + 150860x_n h^3 + 88750x_n^2 h^2 + 20980x_n^3 h + 1715x_n^4)x^3$$

$$- (7543h^3 + 8875x_n h^2 + 3147x_n^2 h + 343x_n^3)x^4 + (8875h^2 + 6294x_n h + 1029x_n^2)x^5$$

$$- (1049h + 343x_n)x^6 + (49)x^7$$

$$W = -(31x_n^6 h + 185x_n^5 h^2 + 535x_n^4 h^3 + 773x_n^3 h^4 + 514x_n^2 h^5 + 120x_n h^6 + 2x_n^7) + (186x_n^5 h + 925x_n^4 h^2 + 2140x_n^3 h^3 + 2319x_n^2 h^4 + 1028x_n h^5 + 120h^6 + 14x_n^6)x$$

$$- \left( (514h^5 + 465x_n^4 h + 1850x_n^3 h^2 + 3210x_n^2 h^3 + 2319x_n h^4 + 42x_n^5) \right) x^2$$

$$+ (773h^4 + 2140x_n h^3 + 1850x_n^2 h^2 + 620x_n^3 h + 70x_n^4)x^3$$

$$- 5(107h^3 + 185x_n h^2 + 93x_n^2 h + 14x_n^3)x^4 + (185h^2 + 186x_n h + 42x_n^2)x^5$$

$$- (31h + 14x_n)x^6 + (2)x^7$$

Evaluating (24) at points  $x = x_{n+6}$  and its derivative at

$x = x_{n+5}, x = x_{n+4}, x = x_{n+3}, x = x_{n+2}, x = x_{n+1}, x = x_{n+1/2}$  yields the following seven discrete hybrid schemes which are used as block integrator;

$$-\frac{968}{220}y_{n+1} + \frac{3025}{579}y_{n+2} - \frac{9680}{1737}y_{n+3} + \frac{6050}{1351}y_{n+4} - \frac{4840}{1737}y_{n+5} + y_{n+6} + \frac{40960}{12159}y_{n+1/2} = \frac{1210}{1737}y_n - \frac{220}{579}h f_{n+6}$$

$$\begin{aligned}
 & -y_{n+1} - \frac{119280}{106785}y_{n+2} + \frac{46172}{106785}y_{n+3} - \frac{15390}{106785}y_{n+4} + \frac{2821}{106785}y_{n+5} + \frac{217088}{106785}y_{n+1/2} = \frac{24626}{106785}y_n - \\
 & \frac{145908}{106785}hf_{n+1} + \frac{168}{106785}hf_{n+6} \\
 & \frac{6510}{60795}y_{n+1} - y_{n+2} - \frac{47096}{6510}y_{n+3} + \frac{11790}{6510}y_{n+4} - \frac{1932}{6510}y_{n+5} - \frac{36864}{6510}y_{n+1/2} = -\frac{6272}{6510}y_n - \\
 & \frac{6510}{574875}hf_{n+2} - \frac{105}{6510}hf_{n+6} \\
 & -\frac{322616}{322616}y_{n+1} + \frac{940800}{322616}y_{n+2} - y_{n+3} - \frac{366075}{322616}y_{n+4} + \frac{45395}{322616}y_{n+5} + \frac{342016}{322616}y_{n+1/2} = \frac{64645}{322616}y_n - \\
 & \frac{729540}{322616}hf_{n+3} + \frac{2100}{322616}hf_{n+6} \\
 & \frac{755520}{703305}y_{n+1} - \frac{981980}{703305}y_{n+2} + \frac{1448048}{703305}y_{n+3} - y_{n+4} - \frac{183848}{703305}y_{n+5} - \frac{488328}{703305}y_{n+1/2} = -\frac{95898}{703305}y_n - \\
 & \frac{729540}{703305}hf_{n+4} - \frac{5880}{703305}hf_{n+6} \\
 & -\frac{458521}{713475}y_{n+1} + \frac{458521}{781200}y_{n+2} - \frac{914340}{458521}y_{n+3} + \frac{930150}{458521}y_{n+4} - y_{n+5} + \frac{471040}{458521}y_{n+1/2} = \frac{96054}{458521}y_n - \\
 & \frac{243180}{458521}hf_{n+5} + \frac{12600}{458521}hf_{n+6} \\
 & -\frac{83679750}{54281216}y_{n+1} + \frac{22799700}{54281216}y_{n+2} - \frac{10588116}{54281216}y_{n+3} + \frac{3779325}{54281216}y_{n+4} - \frac{716870}{54281216}y_{n+5} + y_{n+1/2} = \\
 & -\frac{14124495}{54281216}y_n - \frac{81127040}{54281216}hf_{n+1/2} - \frac{44100}{54281216}hf_{n+6}
 \end{aligned} \tag{25}$$

Equation (25) constitute the members of a zero-stable block integrators of order  $(7,7,7,7,7,7)^T$  with  $C_B = \left[ -\frac{605}{16212}, -\frac{1869}{8}, \frac{951}{8}, -\frac{29355}{16}, \frac{14469}{4}, -\frac{35955}{8}, \frac{2143291}{32} \right]^T$  as the error constants respectively. To start the integration process with  $n=0$ , we use (25) and this produces  $y_1, y_{1/2}, y_2, y_3, y_4, y_5$ , and  $y_6$  simultaneously without the need of any starting method (predictor).

### Stability Analysis

Following Fatunla (1992;1994), that defined the block method to be zero-stable provided the roots  $R_{ij} = 1(1)k$  of the first characteristic polynomial  $\rho(R)$  specified as

$$\rho(R) = \det \left[ \sum_{i=0}^k A^{(i)} R^{k-i} \right] = 0 \tag{26}$$

satisfies  $|R_j| \leq 1$ , the multiplicity must not exceed 2.

The block methods proposed in equations (21) for  $k = 4$  are put in the matrix equation form and for easy analysis the result was normalized to obtain

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

And

$$B^0 = \begin{bmatrix} \frac{242}{225} & -\frac{4807}{5760} & \frac{2197}{5760} & -\frac{1427}{9600} & \frac{151}{5760} \\ \frac{2008}{1575} & -\frac{179}{360} & \frac{119}{360} & -\frac{79}{600} & \frac{59}{2520} \\ \frac{1856}{1575} & \frac{4}{45} & \frac{41}{45} & -\frac{16}{75} & \frac{11}{315} \\ \frac{216}{175} & -\frac{3}{40} & \frac{63}{40} & \frac{51}{200} & \frac{3}{280} \\ \frac{256}{225} & \frac{8}{45} & \frac{52}{45} & \frac{88}{75} & \frac{16}{45} \end{bmatrix}, \quad B^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first characteristic polynomial of the block method is given by

$\rho(R) = \det(RA^0 - A^1)$ . Substituting the  $A^0$  and  $A^1$  into the function above gives

$$\begin{aligned} \rho(R) &= \det \left[ R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \\ &= \det \left[ \begin{bmatrix} R & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \\ &= \det \begin{bmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &= R^5(R(R-1)) - 0 = 0 \\ &R = 0 \text{ (four times) or } R = 1 \end{aligned}$$

The first characteristic polynomial of the block method is given by

$\rho(R) = \det(RA^0 - A^1)$ . Substituting the  $A^0$  and  $A^1$  into the function above gives

$$\rho(R) = \det \left[ R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right]$$

(27)

$$= \det \left[ \begin{bmatrix} R & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] = \det \begin{bmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{bmatrix}$$

$$\Rightarrow R^5(R(R-1)) - 0 = 0$$

$$R_1 = R_2 = R_3 = R_4 = 0 \text{ or } R_5 = 1$$

From (3.0) and equation (27) the hybrid method is zero stable and consistent since the order of the method  $p = 5 > 1$ . And by Henrici (1962); the hybrid method is convergent.

Also, the block methods proposed in equations (25) for  $k = 6$  are put in the matrix equation form and for easy analysis the result was normalized to obtain



$$\begin{aligned}
 &= \det \begin{bmatrix} R & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \det \begin{bmatrix} R & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & R & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & R & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & R-1 \end{bmatrix} \\
 &= R^5(R(R-1)) - 0 = 0
 \end{aligned}$$

$\Rightarrow R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 0$  or  $R_7 = 1$

From (3.0) and equation (29) the hybrid method is zero stable and consistent since the order of the method  $p = 7 > 1$ . And by Henrici (1962); the hybrid method is convergent.

**Numerical Example**

To illustrate the performance of our proposed methods we will compare their performance with exact results. Consider the initial value problem

$y' = \lambda(y - x) + 1, y(0) = 1$

The problem is stiff in nature for negative  $\lambda$  values and it has exact solution  $y(x) = e^{\lambda x} + x$ .

This problem is considered for  $\lambda = -5$ , and  $\lambda = -20$  with step length  $h = 0.01$ . The problem is solved using the Block Hybrid Backward Differentiation Formulae (BHBDF) for  $k = 4$  and  $k = 6$ .

**Table 1:** Proposed (BHBDF) for  $k = 4, \lambda = -5$

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.961229424	0.961229424	5.03E - 10
2	0.02	0.924837418	0.924837418	3.9E - 11
3	0.03	0.890707976	0.890707976	4.27E - 10
4	0.04	0.858730753	0.858730753	8E - 11
5	0.05	0.828800783	0.828800783	7.4E - 11
6	0.06	0.80081822	0.800818221	3.17E - 10
7	0.07	0.774688089	0.77468809	2.8E - 10
8	0.08	0.750320046	0.750320046	3.7E - 11
9	0.09	0.727628151	0.727628152	3.77E - 10
10	0.1	0.706530659	0.706530667	2.86E - 10

**Table 2:** Proposed (BHBDF) for  $k = 4, \lambda = -20$

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.828730753	0.828769259	3.8506896E - 05
2	0.02	0.690320046	0.690432454	1.12408945E - 04
3	0.03	0.578811636	0.578993079	1.81443769E - 04
4	0.04	0.489328964	0.489416887	8.7923593E - 04

5	0.05	0.417879441	0.417989621	<b>1.10180194E - 04</b>
6	0.06	0.361194211	0.361365196	<b>1.70985282E - 04</b>
7	0.07	0.316596963	0.316826495	<b>2.29531969E - 04</b>
8	0.08	0.281896518	0.282023466	<b>1.26948875E - 04</b>
9	0.09	0.255298888	0.255440879	<b>1.4199104E - 04</b>
10	0.1	0.235335283	0.235532264	<b>1.96981522E - 04</b>

**Table 3:** Proposed (BHBDF) for  $k = 6, \lambda = -5$

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.961229424	0.961229424	0
2	0.02	0.924837418	0.924837418	0
3	0.03	0.890707976	0.890707976	0
4	0.04	0.858730753	0.858730753	0
5	0.05	0.828800783	0.828800783	0
6	0.06	0.80081822	0.80081822	0
7	0.07	0.774688089	0.774688089	0
8	0.08	0.750320046	0.750320046	0
9	0.09	0.727628151	0.727628151	0
10	0.1	0.706530659	0.706530659	0

**Table 4:** Proposed (BHBDF) for  $k = 6, \lambda = -20$

N	X	Exact value	Approximate value	Error
0	0.00	1.000000000	1.000000000	0
1	0.01	0.828730753	0.828730765	<b>1.2E - 08</b>
2	0.02	0.690320046	0.690320059	<b>1.3 E - 08</b>
3	0.03	0.578811636	0.578811645	<b>9.0E - 09</b>
4	0.04	0.489328964	0.489328973	<b>9.0E - 09</b>
5	0.05	0.417879441	0.417879445	<b>4. -09</b>
6	0.06	0.361194211	0.361194228	<b>1.7E - 08</b>
7	0.07	0.316596963	0.316596981	<b>1E - 08</b>
8	0.08	0.281896518	0.281896533	<b>1.5 E - 08</b>
9	0.09	0.255298888	0.2552989	<b>1.2E - 08</b>
10	0.1	0.235335283	0.235335293	<b>1.0E - 08</b>

It would be observed from the results in tables 1 and 2 that the error alternates, that's for BHBDF 4 for  $\lambda = -5$  and  $\lambda = -20$  respectively. The BHBDF6 with  $\lambda = -5$  is the best in terms of performance and accuracy as indicated in table 3. In table 4, the BHBDF6 with  $\lambda = -20$  shows an alternate in error, but is of higher accuracy in comparison with results in table 2. However all the block methods produce accurate results when compared with exact results.

### Conclusions

We have derived the hybrid form of the Backward Differentiation Formulae (BDF) for  $k = 4$  and  $k = 6$ . The idea of Multistep Collocation (MC) was used to reformulate the derived hybrid formulae into continuous form which were employed as block methods for direct solution of  $y' = f(x, y)$ . A convergence analysis of the discrete hybrid methods to establish their

effectiveness and reliability is presented. The methods were tested on some stiff IVP and shown to perform satisfactorily without the requirement of any starting method.

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